# GEOMETRY AND INTERSECTION THEORY ON HILBERT SCHEMES OF FAMILIES OF NODAL CURVES

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ABSTRACT. We study the relative Hilbert scheme of a family of nodal (or smooth) curves, over a base of arbitrary dimension, via its (birational) *cycle map*, going to the relative symmetric product. We show the cycle map is the blowing up of the discriminant locus, which consists of cycles with multiple points. We work out the action of the blowup or 'discriminant' polarization on some natural cycles in the Hilbert scheme, including generalized diagonals and cycles, called 'node scrolls', parametrizing schemes supported on singular points. We derive an intersection calculus for Chern classes of tautological vector bundles, which are closely related to enumerative geometry.

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Consider a family of curves given by a flat projective morphism

$$\pi : X \to B$$

over an irreducible base, with fibres

$$X_b = \pi^{-1}(b), b \in B$$

which are irreducible nonsingular for the generic $b$ and at worst nodal for every $b$. For example, $X$ could be the universal family of automorphism-free curves over the appropriate open subset of $\overline{\mathcal{M}}_g$, the moduli space of Deligne-Mumford stable curves. Many questions in the classical projective and enumerative geometry of this family can be naturally phrased, and in a formal sense solved (see for instance [12]), in the context of the *relative Hilbert scheme*

$$X_B^{[m]} = \mathrm{Hilb}_m(X/B),$$

which parametrizes length-$m$ subschemes of $X$ contained in fibres of $\pi$, and the natural *tautological vector bundle* $\Lambda_m(E)$, living on $X_B^{[m]}$, that is associated to some vector bundle $E$ on $X$ (for example, the relative dualizing sheaf $\omega_{X/B}$). Typically, the questions include ones involving relative multiple points and multisecants in the family, and the formal solutions involve Chern numbers of the tautological bundles. Thus, turning these formal solutions into meaningful ones requires computing the Chern numbers in question. Aside from some low-degree cases, this problem was left open in [12]. Our main purpose here is to solve this problem in general. More than that, we shall in fact provide a calculus to compute arbitrary polynomials in the Chern classes of the tautological bundles. In the 'absolute' case $E = \omega_{X/B}$, the computation ultimately reduces these polynomials to polynomials in Mumford's tautological classes [7] on various boundary strata of $B$. Fortunately these have been computed by Kontsevich [4] following a conjecture by Witten. Note that the boundary naturally carries families of *pointed* curves (of lower genus), the points being node preimages, and the tautological classes involved will include the cotangent or $\psi$ classes on these.

The calculus that we develop is fundamentally *recursive* in $m$. The recursion involves *flag Hilbert schemes*, in particular the full-flag scheme $W^m = W^m(X/B)$ studied in [12], as well as its 'flaglet' analogue $X_B^{[m,m-1]}$, parametrizing flags of schemes of lengths $m, m-1$. Its starting point is an analogue for $X_B^{[m,m-1]}$ of the *splitting principle*, established in [12]. This result (Corollary 3.5 ) expresses the total Chern class $c(\Lambda_m(E))$, pulled back to $X_B^{[m,m-1]}$, as a product of $c(\Lambda_{m-1}(E))$ and a simple 'discriminant' factor involving Chern classes of $E$ and a certain *discriminant* divisor $\Gamma^{(m)}$. In order to compute polynomials in the Chern classes of $\Lambda_m(E)$, we are thus reduced recursively to studying the multiplication action of powers of $\Gamma^{(m)}$ on polynomials in $c(\Lambda_{m-1}(E))$.

Among other things, the Splitting Principle suggests the central role played by $\Gamma^{(m)}$ in the study of the Hilbert scheme $X_B^{[m]}$, stemming from the fact that it effectively encodes the information contained in $X_B^{[m]}$ beyond the relative symmetric product $X_B^{(m)}$. The latter viewpoint is further supported by the *Blowup Theorem* 1.1 that we prove below, which says that via the cycle (or 'Hilb-to-Chow') map

$$\mathfrak{c}_m : X_B^{[m]} \to X_B^{(m)},$$

the Hilbert scheme is equivalent to the blowing up of the *discriminant locus*

$$D^m \subset X_B^{(m)},$$

which is the Weil divisor parametrizing nonreduced cycles, and where $\Gamma^{(m)} = \mathfrak{c}_m^{-1}(D^m)$, so that $-\Gamma^{(m)}$ can be identified with the natural $\mathcal{O}(1)$ polarization of the blowup. The Blowup Theorem is valid without dimension restrictions on $B$.

Given the Blowup Theorem, our intersection calculus proceeds along the following lines suggested by the aforementioned Splitting Principle. On each Hilbert scheme $X_B^{[m]}$, we identify a collection of geometrically-defined *tautological classes* which, together with base classes coming from $X$, additively generate what we call the *tautological module* $T^m = T^m(X/B)$. These classes come in two main flavors.

- The (relative) *diagonal classes*: these are loci of various codimensions defined by diagonal conditions pulled back from the relative symmetric product of $X/B$, possibly twisted by base classes; they are analogous to the 'creation operators' in Nakajima's work in the case of smooth surfaces.
- The *node classes*: these are associated to relative nodes $\theta$ of $X/B$, hence roughly to boundary components, and come in 2 kinds: the *node scrolls*, which parametrize schemes with a length $> 1$ component at $\theta$, and are $\mathbb{P}^1$-bundles over a relative Hilbert scheme associated to the normalization along $\theta$ of the boundary subfamily of $X/B$ where $\theta$ lives; and the *node sections*. which are simply (intersection) products of a node scroll by the discriminant $\Gamma^{(m)}$.

Then the first main component of the calculus is the *Module Theorem* 2.1, which says that $T^m$ is (computably!) a module over the polynomial ring $\mathbb{Q}[\Gamma^{(m)}]$. Included in this is the nontrivial assertion that $\mathbb{Q}[\Gamma^{(m)}] \subset T^m$; this means we can compute, recusively at least, arbitrary powers of $\Gamma^{(m)}$ as $\mathbb{Q}$-linear combinations of tautological classes.

Rounding out the story is the *Transfer Theorem* 3.3, which computes the transfer (pull-push) operation on tautological classes from $X_B^{[m-1]}$ to $X_B^{[m]}$ via the flaglet Hilbert scheme $X_B^{[m,m-1]}$, viewed as a correspondence.

The conjunction of the Splitting Principle, Module Theorem and Transfer Theorem computes all polynomials in the Chern classes, in particular the Chern numbers, of $\Lambda_m(E)$ as $\mathbb{Q}$-linear combinations of tautological classes on $X_B^{[m]}$.

Note that if $X$ is a smooth surface, there is a natural closed embedding

$$j_\pi^{[m]} : X_B^{[m]} \subset X^{[m]}$$

of the relative Hilbert scheme in the full Hilbert scheme of $X$, which is a smooth projective $2m$-fold. There is a large literature on Hilbert schemes of smooth surfaces and their cohomology and intersection theory, due to Ellingsrud-Strømme, Göttsche, Nakajima, Lehn and others, see [3, 5, 6, 8] and references therein. In particular, Lehn [5] gives a formula for the Chern classes of the tautological bundles on the full Hilbert scheme $X^{[m]}$, from which one can derive a formula for the analogous classes on $X_B^{[m]}$ if $X$ is a smooth surface, but this does not, to our knowledge, yield Chern numbers (besides the top one) on $X^{[m]}$, much less $X_B^{[m]}$ (the two sets of numbers are of course different). Going from Chern *classes* to Chern *numbers* it a matter of working out the top-degree multiplicative structure, i.e. the intersection calculus. When $X$ is a surface with trivial canonical bundle, Lehn and Sorger [6] have given a rather involved description of the mutiplicative structure on the cohomology of $X^{[m]}$ in all degrees, not just the top one. While products on $X^{[m]}$ and $X_B^{[m]}$ are compatible $j_\pi^{[m]}$, it's not clear how to compute intersection products, especially intersection numbers on $X_B^{[m]}$ from products on $X^{[m]}$, even in case $X$ has trivial canonical bundle. Indeed some of our additive generators directly involve the fibre nodes of the family $X/B$ and do not appear to come from classes on $X^{[m]}$. However, note that the class of surfaces with trivial canonical bundle that fibre (via a morphism, not a rational map) over a smooth curve is very small (and even smaller if one assumes at least one singular fibre), so the potential intersection between our work and [6] is very small. Besides, our calculus works for a higher dimensions as well.

The paper is organized as follows. A preliminary §0 defined certain combinatorial numbers to be needed later. Then Chapter 1 is devoted to the proof of the Blowup Theorem. Actually it is the proof, rather than the statement, of the Theorem, that is of principal interest to us. The proof proceeds by first constructing

an explicit model $H_m$ for the cycle map, locally over the base in a neighborhood of a cycle of the form the point $m[p]$ where $p$ is a relative node (this cycle is an explicit, albeit non $\mathbb{Q}$-Gorenstein lci-quotient singularity on the relative symmetric product); then, via a 'reverse engineering' process, we identify the cycle map as the blowup of the discriminant locus (this without advance knowledge of the ideal of the latter). In the sequel, our main use of the Blowup Theorem is as a convenient way of gluing the $H_n$ models, $n \leq m$ together, globally over the base $X_B^{(m)}$.

Chapter 2 is devoted to the definition of the Tautological Module $T^m$ and proof of the Module Theorem. As a convenient artifice, we first define the appropriate classes on an ordered model of the Hilbert scheme and subsequently pass to the quotient by the symmetrization map.

In Chapter 3 we study the geometry of the flaglet Hilbert scheme $X_B^{[m,m-1]}$, largely referring to [13], and derive properties of the transfer operation. We then review the splitting principle form [12], which enables us to complete our clacu-lus.

For a detailed sketch of the proof of Theorem 1 (though without all the details), and some other applications, see [11].

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## 0. Preliminaries

0.1. **Staircases.** We define a combinatorial function that will be important in computations to follow. Denote by $Q$ the closed 1st quadrant in the real $(x, y)$ plane, considered as an additive cone. We consider an integral *staircase* in $Q$. Such a staircase is determined by a sequence of points

$$(0, y_m), (x_1, y_m), (x_1, y_{m-1}), (x_2, y_{m-1}), ..., (x_m, y_1), (x_m, 0)$$

where $0 < x_1 < ... < x_m, 0 < y_1 < ... < y_m$, are integers, and consists of the polygon

$$B = (-\infty, x_1) \times \{y_m\} \bigcup \{x_1\} \times [y_{m-1}, y_m] \bigcup [x_1, x_2] \times \{y_{m-1}\} \bigcup ... \bigcup \{x_m\} \times (-\infty, y_1].$$

The *upper region* of $B$ is by definition

$$R = B + Q = \{(b_1 + u_1, b_2 + u_2) : (b_1, b_2) \in B, u_1, u_2 \geq 0\}$$

We call such $R$ a *special infinite polygon*. The closure of the complement

$$S = R^c := \overline{Q \setminus R} \subset Q$$

has finite (integer) area and will be called a *special finite polygon*; in fact the area of $S$ coincides with the number of integral points in $S$ that are $Q$-interior, i.e. not in $R$; these are precisely the integer points $(a, b)$ such that $[a, a+1] \times [b, b+1] \subset S$.

Note that we may associate to $R$ a monomial ideal $\mathcal{I}(R) < \mathbb{C}[x, y]$ generated by the monomials $x^a y^b$ such that $(a, b) \in R \cap Q$. The area of $S$ then coincides with $\dim_{\mathbb{C}}(\mathbb{C}[x, y]/\mathcal{I}(R))$. It is also possible to think of $S$ as a partition or Young tableau, with $x_1$ many blocks of size $y_m$, ..., $(x_{i+1} - x_i)$ many blocks of size $y_{m-i}$...

Fixing a natural number $m$, we define the *basic special finite polygon associated to* $m$ as

$$S_m = \bigcup_{i=1}^{m} [0, \binom{m-i+1}{2}] \times [0, \binom{i+1}{2}].$$

It has area

$$\alpha_m = \sum_{i=1}^{m-1} i \binom{m+1-i}{2} = \frac{m(m+2)(m^2-1)}{24}$$

and associated special infinite polygon denoted $R_m$. Now for each integer $j = 1, ..., m-1$ we define a special infinite polygon $R_{m,j}$ as follows. Set

$$P_j = (m-j, -j) \in \mathbb{R}^2,$$

$$R_{m,j} = R_m \cup (R_m + P_j) \cup [0, \infty) \times [j, \infty)$$

(where $R_m + P_j$ denotes the translate of $R_m$ by $P_j$ in $\mathbb{R}^2$). We also define $P_j^-, P_j^+$ analogously with $m-j$ replaced by $m-j-1$ and $m-j+1$, respectively. Then let $S_{m,j} = R_{m,j}^c$,

$$\beta_{m,j} = \text{area}(S_{m,j}),$$

$$\beta_m = \sum_{j=1}^{m-1} \beta_{m,j}.$$

We also define $\beta_{m,j}^{\pm}$ based on $P_j^+, P_j^-$ respectively. It is easy to see that

(0.1.1) $$\beta_{m,1} = \binom{m}{2}, \beta_{m,2} = \binom{m}{2} + \binom{m-1}{2} - 1, \beta_{m,j} = \beta_{m,m-j}$$

,

(0.1.2) $$\beta_{m,1}^- = \binom{m-1}{2} - 1,$$

but otherwise we don't know a closed-form formula for these numbers in general. A few small values are

$$\beta_{2,1} = \beta_2 = 1$$

$$\vec{\beta}_3 = (3, 3), \beta_3 = 6$$

$$\vec{\beta}_4 = (6, 8, 6), \beta_4 = 20$$

$$\vec{\beta}_5 = (10, 15, 15, 10), \beta_5 = 50$$

$$\vec{\beta}_6 = (15, 24, 27, 24, 15), \beta_6 = 105.$$

For example, for $m = 5$ the relevant finite polygons, viewed as partitions, are

$$S_{5,1} = 1^{10}, S_{5,2} = 2^6 1^3, S_{5,3} = 3^5, S_{5,4} = 3^2 2^2.$$

Set

$$J_m = \mathcal{I}(R_m).$$

For an interpretation of the $\beta_{m,j}$ as exceptional multiplicities associated to the blowup of the monomial ideal $J_m$, see §1.6 below.

## 0.2. **Products, diagonals, partitions.**

0.2.1. *Partitions, distributions and shapes.* The intersection calculus we aim to develop is couched in terms certain diagonal-like loci on products, defined in the general case in terms of partitions. To facilitate working with these loci systematically, we now establish some conventions, notations and simple remarks related to partitions.

Given a natural number $m$, a *partition* in $[1, m]$ for us is a sequence of pairwise disjoint index sets

$$I_1, ..., I_r \subset [1, m] \cap \mathbb{Z}.$$

Elsewhere this is sometimes called a 'labelled partition'. Two partitions are said to be *equivalent* if they differ only by singleton blocks and by renumbering of blocks. A partition in $[1, m]$ is said to be *full* if $\bigcup_{\ell} I_\ell = \{1, ..., m\}$, *irredundant* if it has no singleton blocks. Clearly any partition is equivalent to a full one with nonincreasing block cardinalities, and a to a irredundant one, again with nonincreasing block cardinalities.

We partially order the partitions by declaring that $\Phi_1 \prec \Phi_2$ if every non-singleton block of $\Phi_1$ is a union of (possibly singleton) blocks of $\Phi_2$. Thus $\Phi_1$ and $\Phi_2$ are equivalent iff $\Phi_1 \prec \Phi_2$ and $\Phi_2 \prec \Phi_1$ .

Now by a *length distribution* (elsewhere called a partition) we mean a function $\underline{n}$ of finite support from the positive integers to the nonnegative integers. The *total length* of $\underline{n}$ is by definition $\sum_{\ell} \underline{n}(\ell)$. To any partition $(I.)$ there is an associated length distribution $\underline{n} = (|I.|)$, defined by $\underline{n}(\ell) = |I_\ell|, \forall \ell$. In light of these, there are natural notions of fullness (with respect to $m$), precedence and equivalence for distributions.

Now given a distribution $\underline{n}$, let $n_1 > n_2 > .., n_r \geq 1$ be its set of distinct nonzero values, i.e. the *value sequence*, and

$$\mu_\ell = |\{\ell : \underline{n}(\ell) = n\}|$$

be its *frequency sequence* (also called sometimes frequency function using the notation $\mu_{\underline{n}}$). The *shape* of $\underline{n}$ is the (obviously finite) sequence

(0.2.3) $$(n.^{(\mu.)}) = (n^{(\mu(n))}) = (..., m^{(\mu_{\underline{n}}(m))}, ..., 1^{(\mu(n.)(1))}) = (n_1^{(\mu_1)}, ..., n_r^{(\mu_r)})$$

where the exponents are taken formally with respect to the juxtaposition operation (and all terms with zero exponent in the LHS are omitted). The shape of $\underline{n}$ obviously determines $\underline{n}$, and indeed distribution and shape are equivalent data, each preferable in different situations, and will be used interchangeably. A distribution is in *loose form* (resp. *shape form*) if it is written as $(n_1 \geq n_2 \geq ...)$ (resp. $(n_1^{(\mu_1)}, ...)$) where each $n_i$ is the $i$-th member (resp. the $i$-th distinct member) of it, in nonincreasing (resp. strictly decreasing) order.

The shape of a partition $(I.)$, i.e. the shape of its distribution, is also written $(|I.|)$ and has the form $(n_1^{\mu_{(I.)}(n_1)}, ..., n_r^{\mu_{(I.)}(n_r)})$ where the $n_\ell, \mu_\ell = \mu_{(I.)}(n_\ell)$ are precisely the heights and widths of the adjacent rectangles forming the Young tableau for $(I.)$. Thus we may think of a shape (or a distribution) as an unspecified partition having the given shape or distribution.

Next we define some natural operations on distributions that we will need. If $(n'.), (n".)$ are distributions, we can define a new distribution denoted $(n'.)\coprod(n".)$ by the condition that its frequency function $\mu$ is the sum of that of $(n'.)$ and $(n".)$, i.e.

(0.2.4) $$\mu_{(n'.)\coprod(n".)} = \mu_{(n'.)} + \mu_{(n".)}.$$

This corresponds to the operation of disjoint union of partitions. For a distribution $(n.)$ and an integer $k$, we define a distribution $(n.) \setminus k$ by

(0.2.5) $$\mu_{(n.)\setminus k} = \mu_{(n.)} - \mathbf{1}_k$$

where corresponds to removing a block of size $k$ from a partition. By convention, $(n.) \setminus k = \emptyset$ if $(n.)$ has no block of size $k$; more generally, a distribution of shape is considered empty of the corresponding frequency function has a negative value.

We define another operation $(n.)^{-\ell}$ as follows: let $n_1 > ... > n_\ell > ...$ be the distinct block sizes occurring in $(n.)$. Then

(0.2.6) $$(n.)^{-\ell} = (n.) \setminus n_\ell \coprod (n_\ell + 1)$$

which corresponds to removing a block of size $n_\ell$ and replacing it by one of size $n_\ell + 1$. Also,

(0.2.7) $$u_{j,\ell}(n.) = (n.) \setminus n_j \setminus n_\ell \coprod (n_j + n_\ell)$$

which corresponds to uniting an $n_j$ and an $n_\ell$ block.

Define *multidistribution data* $\phi$ as $(n_\ell : \underline{n}'|\underline{n}")$ where $(\underline{n}')\coprod(\underline{n}") = (\underline{n})\setminus n_\ell$, where $(\underline{n})$ is a (usually full) distribution on $[1, m]$ (this notation, motivated by the partition case, indicates removing a single block of size $n_\ell$ and breaking up the remainder into $x$ type and $y$ type).

0.2.2. *Products and diagonal loci.* Now given any set $X$ and partition $(I.)$, we define the diagonal locus $X^{(I.)}$ as the set of 'locally constant', i.e. constant on

blocks, functions $\bigcup I. \to X$. This is called the (ordered) diagonal locus corresponding to $(I.)$. If $X$ is endowed with a map to $B$, there is an analogous relative notion $X_B^{(I.)}$ referring to functions such that the composite $\bigcup I. \to B$ is constant (the extension from the absolute to the relative case involves no new ideas and will not be emphasized in what follows). The diagonal locus $X^{(I.)}$ is a subset (closed subscheme, if $X$ is a separated scheme) of the cartesian product $X^m$, and can be identified (isomorphically, if $X$ is a scheme), in terms of the above shape notation, with $\prod_{\ell=1}^{r} X^{\mu_\ell}$.

Now there is an unordered analogue of $X^{(I.)}$, depending only on the shape of $I.$ and denoted $X^{\underline{n}}$ or $X^{(n.^{\mu.})}$ or $X^{(\coprod n^{\mu(n)})}$. This is called a symmetric diagonal locus. It coincides with the image of $X^{(I.)}$ in the symmetric product $X^{(m)}$, and can in turn be identified with $\prod_{\ell=1}^{r} X^{(\mu_\ell)} = \prod_{n=\infty}^{1} X^{(\mu(n))}$. Note the natural 'diagonal' embedding

(0.2.8) $$\delta_{n.,\ell} : X^{(n.-\ell)} \to X^{(n.)} \times X$$

Next, multidistribution data have to do with a situation where X comes equipped with a representation $X = X' \cup X"$ ; the case $X' = X"$ will require special treatment. The corresponding (symmetric) diagonal locus is

(0.2.9) $$X^{\phi} = \prod_{\ell} (X')^{(\mu'_\ell)} \times \prod_{\ell} (X")^{(\mu"\ell)}$$

where $\underline{n}' = (n'.^{\mu.'}), \underline{n}" = (n".^{\mu."})$. We view $X^{\phi}$ as embedded in $(X')^{(n')} \times (X")^{(n")}$, where $n' = |\underline{n}'|, n" = |\underline{n}"|$, the embedding coming from by the various diagonal embeddings $X' \to (X')^{(n.)}$ which induce $(X')^{(\mu.)} \to ((X')^{(n'.)})^{(\mu.)}$; similarly for $X"$. Note the natural map $X^{\phi} \to X^{(n'+n")}$.

If $X' = X"$, we take $n" = \emptyset$.

Now, an obvious issue that comes up is to determine the degree of the symmetrization map $X^{(I.)} \to X^{(|I.|)}$. To this end, let $I.$ be a full partition on $[1, m]$ with length distribution $(n.)$. Let

$$in(I.) \lhd out(I.) (< \mathfrak{S}_m)$$

denote the groups of permutations of $\{1, ..., m\}$ taking each block of $I.$ to the same (resp. to some) block. Then

$$aut(I.) = out(I.)/in(I.)$$

is the 'automorphism group' of $(I.)$ (block permutations induced by elements of $\mathfrak{S}_m$). Let

$$a(I.) = |aut(I.)|.$$

Then $a(I.)$ is easily computed: if $(n.^{\mu.})$ is the shape of $(I.)$, we have

(0.2.10) $$a(I.) = \prod (\mu_\ell)! \quad .$$

Clearly $a(I.)$ depends only on the length distribution $\underline{n} = (|I.|)$, so we may (abusively) write $a(\underline{n})$ for $a(I.)$. Significant for our purposes, but easy to verify, is the following

**Lemma 0.1.** *If* $I.$ *is full and* $\underline{n} = (|I.|)$ *, then the mapping degree*

$$\deg(X^{(I.)} \to X^{(|I.|)}) = a(\underline{n}). \tag{0.2.11}$$

□

0.2.3. *Cohomology and base classes.* Finally, we briefly discuss cohomology. If $X$ is a reasonable space (topological, scheme, etc.) and $H^{\cdot}$ is a reasonable $\mathbb{Q}$-valued cohomology theory (singular, Chow, etc.), then we have a homomorphism

$$\mathrm{Sym}^{\mu}(H^{\cdot}(X)) \to H^{\cdot}(X^{(\mu)}), \tag{0.2.12}$$

where $X^{(\mu)}$ is the appropriate symmetric product, whose image is called the ring of *base classes* on $X^{(\mu)}$ and denoted $B(X^{(\mu)})$ (abusively so, of course, since it depends on $X$). Similarly, in the relative situation $X/B$, we get a ring homomorphism ('symmetrization')

$$\sigma : \mathrm{Sym}^{\mu}(H^{\cdot}(X)) \to H^{\cdot}(X^{(\mu)}_B), \tag{0.2.13}$$

whose image is called the ring of base classes on $X^{(\mu)}_B$ and denoted $B(X^{(\mu)}_B)$. The ring $B(X^{(\mu)})$ is clearly generated by images, called *polyclasses* of classes of the form

$$X^{(\mu)}[\alpha^{(\lambda)}] := \sigma(\alpha^{\lambda}), \lambda \leq \mu. \tag{0.2.14}$$

When $\lambda = 1$, the superscript will be omitted. We also write

$$X^{(\mu)}[\alpha_1 \cdots \alpha_{\lambda}] := \sigma(\alpha_1 \cdots \alpha_{\lambda}), \tag{0.2.15}$$

this being simply

$$\frac{1}{\lambda!}\pi_*(\sum_{s\in\mathfrak{S}_{\lambda}} \alpha_{s(1)} \boxtimes \cdots \alpha_{s(\lambda)} \boxtimes 1 \cdots \boxtimes 1)$$

where $\pi : X^{\mu} \to X^{(\mu)}$ is the natural map and $\boxtimes$ refers to exterior cup product (restricted on the fibred product). Thus,

$$X^{(\mu)}[\alpha^{(\lambda)}] = X^{(\mu)}[\alpha^{(\lambda)}1^{(\mu-\lambda)}].$$

We will primarily use the natural analogues of these constructions in the relative case $X/B$. Note that the polyclasses are multiplied according to the rule

$$X^{(\mu)}[\alpha_1^{(\lambda_1)}]X^{(\mu)}[\alpha_2^{(\lambda_2)}] = \sum_{\nu=0}^{\min(\lambda_1,\lambda_2)} X^{(\mu)}[\alpha_1^{(\lambda_1-\nu)} \cdot \alpha_2^{(\lambda_2-\nu)} \cdot (\alpha_1 {}_{\dot{X}} \alpha_2)^{(\nu)}] \tag{0.2.16}$$

Given spaces $X_1, ..., X_r$ as above, we write

$$(X_1 \times ... \times X_r)[\alpha_1, ..., \alpha_r] = \alpha_1 \boxtimes \cdots \boxtimes \alpha_r.$$

These are called base classes on the Cartesian product. In particular, given a distribution $(n.^{\mu.})$, we obtain in this way classes $X^{(n.^{\mu.})}[\alpha_1^{(\lambda_1)}, ..., \alpha_r^{(\lambda_r)}]$; these are again called *base classes* on the symmetric diagonal locus $X^{(n.^{\mu.})} \simeq \prod_{i=1}^{r} X^{(\mu(n_i))}$.

We will require some operations on these base classes. Consider a base class $\gamma$ which we write in the form

$$\gamma = X^{(n.^{\mu.})}[\coprod \alpha(n)^{(\lambda(n))}]. \tag{0.2.17}$$

Then generally, given any (co)homology vector $\beta. = (\beta_1, ..., \beta_r)$, with each $\beta_i$ a class on $X_i$, we can define a new (co)homology class by

$$\gamma \star_t [\beta.] = \gamma \cup s_t(\beta) \tag{0.2.18}$$

where $s_t$ is the $t$-th elementary symmetric function (in terms of the exterior $\boxtimes$ product, where missing factors are deemed $= 1$). A particular case we will use is

$$X^{(n.^{\mu.})}[\alpha_1^{(\lambda_1)}, ..., \alpha_r^{(\lambda_r)}] \star_1 [\omega_1, ..., \omega_r] = \sum_{\ell=1}^{r} X^{(n.^{\mu.})}[\alpha.^{(\lambda.)}] \cup p_\ell^*(\omega_\ell) \tag{0.2.19}$$

For a single class $\omega_\ell$ on $X$, set

$$X^{(n.^{\mu.})}[\alpha_1^{(\lambda_1)}, ..., \alpha_r^{(\lambda_r)}] \star_{1,\ell} [\omega_\ell] = X^{(n.^{\mu.})}[\alpha.^{(\lambda.)}] \cup p_\ell^*(\omega_\ell) \tag{0.2.20}$$

Similar notations will be used in the case of multipartition data, e.g.

$$\begin{gathered} X^{(n:n'.^{(\mu'.)}|n".^{(\mu".)})}[(\alpha_1)^{(\lambda'_1)}, ...; (\alpha"_1)^{(\lambda"_1)}, ...] = \\ (X')^{(n'.^{(\mu'.)})}[(\alpha_1)^{(\lambda'_1)}, ...] \times (X")^{(n".^{(\mu".)})}[(\alpha"_1)^{(\lambda"_1)}, ...] \end{gathered} \tag{0.2.21}$$

Again, all these constructions also have natural analogues in the relative situation.

Also, there are analogues of these constructions with $H^{\cdot}(X)$ replaced by any $\mathbb{Q}$-subalgebra of itself. For example, $X$ may be a surface fibred over a smooth curve $B$ and endowed with a polarization $L$, in which case we will usually consider the subalgebra

$$K^{\cdot}(X/B) = H^{\cdot}(B)[L, \omega_{X/B}]. \tag{0.2.22}$$

0.2.4. *Canonical class and half-discriminant.* Let $X/B$ be a family of smooth curves and

$$D^m = X_B^{(2,1^{m-2})} \subset X_B^{(m)}$$

the big diagonal or discriminant. This is a reduced Cartier divisor, defined locally by the discriminant function which is a polynomial in the elementary symmetric functions of a local parameter of $X/B$. The associated line bundle $\mathcal{O}(D^m)$ is always divisible by 2 as line bundle. One way to see this is to note that $D^m$ is the branch locus of a flat double cover

$$\epsilon : X_B^{\{m\}} \to X_B^{(m)} \tag{0.2.23}$$

where $X_B^{\{m\}} = X_B^m/\mathfrak{A}_m$ is the 'orientation product', generically parametrizing an $m$-tuple together with an orientation. An explicit 'half' of $\mathcal{O}(D^m)$ is given by

$$h = X_B^{(m)}[\omega_{X/B}] \otimes \omega^{-1}_{X_B^{(m)}/B} \tag{0.2.24}$$

Indeed $\epsilon^* h$ is precisely the (reduced) ramification divisor of $\epsilon$, which is half of $\epsilon^* D^m$. In particular, note that $\epsilon^* h$ is effective. We also have

$$\epsilon_* \mathcal{O}_{X_B^{\{m\}}} = \mathcal{O}_{X_B^{(m)}} \oplus h^{-1}. \tag{0.2.25}$$

## 1. The cycle map as blowup

1.1. **Set-up.** Let

$$\pi : X \to B$$

be a family of nodal (or smooth) curves with $X, B$ smooth. Let $X_B^m, X_B^{(m)}$, respectively, denote the $m$th Cartesian and symmetric fibre products of $X$ relative to $B$. Thus, there is a natural map

$$\omega_m : X_B^m \to X_B^{(m)}$$

which realizes its target as the quotient of its source under the permutation action of the symmetric group $\mathfrak{S}_n$. Let

$$\mathrm{Hilb}_m(X/B) = X_B^{[m]}$$

denote the relative Hilbert scheme paramerizing length-$m$ subschemes of fibres of $\pi$, and

$$\mathfrak{c} = \mathfrak{c}_m : X_B^{[m]} \to X_B^{(m)}$$

the natural *cycle map* (cf. [1]). Let $D^m \subset X_B^{(m)}$ denote the discriminant locus or 'big diagonal', consisting of cycles supported on $< m$ points (endowed with the reduced scheme structure). Clearly, $D^m$ is a prime Weil divisor on $X_B^{(m)}$, birational to $X \times_B \mathrm{Sym}^{m-2}(X/B)$, though it is less clear what the defining equations of $D^m$ on $X_B^{(m)}$ are near singular points. The main purpose of Part 1 is to prove

**Theorem 1.1** (Blowup Theorem)**.** *The cycle map*

$$\mathfrak{c}_m : X_B^{[m]} \to X_B^{(m)}$$

*is the blow-up of $D^m \subset X_B^{(m)}$.*

The proof will proceed through an explicit construction of the map $\mathfrak{c}_m$, source included, locally over $X_B^{(m)}$. This construction will play a crucial role in the entire paper and therefore will be studied in greater detail than is required solely to prove Theorem 1.1.

1.2. **Preliminary reductions.** To begin with, we reduce the Theorem to a local statement over a neighborhood of a 1-point cycle $mp \in X_B^{(m)}$ where $p \in X$ is a node of $\pi^{-1}(\pi(p))$. Set

$$\Gamma^{(m)} = \mathfrak{c}_m^{-1}(D^m) \subset X_B^{[m]}.$$

It was shown in [12], and will be reviewed below, that $\mathfrak{c}_m$ is a small birational map (with fibres of dimension $\leq \min(m/2, \max\{|\text{sing}(X_b)|, b \in B\})$), all its fibres (aka punctual Hilbert schemes or products thereof) are reduced, and that $X_B^{[m]}$ is smooth.

**Lemma 1.2.** $\Gamma^{(m)}$ *has no embedded components.*

We defer the proof to §1.5 below.

Assuming the Lemma, $\Gamma^{(m)}$ is an integral, automatically Cartier, divisor, and therefore $\mathfrak{c} = \mathfrak{c}_m$ factors through a map $\mathfrak{c}'$ to the blow-up $B_{D^m}(X_B^{(m)})$, and it would suffice to show that $\mathfrak{c}'$ is an isomorphism, which can be checked locally. This is the overall plan that we now set out to execute. First a few reductions.

Let $U \subset X_B^{(m)}$ denote the open subset consisting of cycles having multiplicity at most 1 at each fibre node. Then $U$ is smooth and the cycle map $\mathfrak{c}_m : \mathfrak{c}_m^{-1}(U) \to U$ is an isomorphism. Consequently, it will suffice to show $\mathfrak{c}_m$ is equivalent to the blowing-up of $D^m$ locally near any cycle $Z \in X_B^{(m)}$ having degree $> 1$ at some point of the locus $X^\sigma \subset X$ of singular points of $\pi$ (i.e. singular points of fibres).

Our next point, a standard one, is that it will suffice to analyze $\mathfrak{c}_m$ locally over a neighborhood of a 'maximally singular' fibre, i.e. one of the form $mp$ where $p$ is a singular point of $\pi$. For a cycle $Z \in X_B^{(m)}$, we let $(X_B^{(m)})_{(Z)}$ denote its open neighborhood consisting of cycles $Z'$ which are 'no worse' than $Z$, in the sense that their support $\text{supp}(Z')$ has cardinality at least equal to that of $\text{supp}(Z)$. Similarly, for a $k$-tuple $Z. = (Z_1, ..., Z_k) \in \prod X_B^{(m_i)}$, we denote by $(\prod X_B^{(m_i)})_{(Z.)}$ its open neighborhood in the product consisting of 'no worse' multicycles, i.e. $k$-tuples $Z'. \in \prod X_B^{(m_i)}$ such that each $Z'_i$ is no worse than $Z_i$ and the various $Z'_i$ are mutually pairwise disjoint. We also denote by $(X_B^{[m]})_{(Z)}, (\prod X_B^{[m_i]})_{(Z.)}$ the respective preimages of the no-worse neighborhoods of $Z$ and $Z.$ via $\mathfrak{c}_m$ and $\prod \mathfrak{c}_{m_i}$. Now writing a general cycle

$$Z = \sum_{i=1}^{k} m_i p_i$$

with $m_i > 0, p_i$ distinct, and setting $Z_i = m_i p_i$, we have a cartesian (in each square) diagram

$$\begin{array}{ccc}
(\prod\limits_{i=1}^{k} {}_B\ X_B^{[m_i]})_{(Z.)} & \xrightarrow{\prod \mathfrak{c}_{m_i}} & (\prod\limits_{i=1}^{k} {}_B\ X_B^{(m_i)})_{(Z.)} \\
e_1 \uparrow & \square & \uparrow d_1 \\
H & \rightarrow & S \\
e \downarrow & \square & \downarrow d \\
(X_B^{[m]})_{(Z)} & \xrightarrow{\mathfrak{c}_m} & (X_B^{(m)})_{(Z)}
\end{array}$$

Here $H$ is the restriction of the natural inclusion correspondence on Hilbert schemes:

$$H = \{(\zeta_1, ..., \zeta_k, \zeta) \in (\prod_{i=1}^{k} {}_B\ X_B^{[m_i]})_{(Z.)} \times (X_B^{[m]})_{(Z)} : \zeta_i \subseteq \zeta, i = 1, ..., k\},$$

and similarly for $S$. Note that the right vertical arrows $d, d_1$ are étale and induce analytic isomorphisms between some analytic neighborhoods $U$ of $Z$ and $U'$ of $Z.$ and the left vertical arrows $e, e_1$ are also étale and induce isomorphisms between $\mathfrak{c}_m^{-1}(U)$ and $(\prod \mathfrak{c}_{m_i})^{-1}(U')$.

Now by definition, the blow-up of $X_B^{(m)}$ in $D^m$ is the Proj of the graded algebra

$$A(\mathcal{I}_{D^m}) = \bigoplus_{n=0}^{\infty} \mathcal{I}_{D^m}^n.$$

Note that

$$d^{-1}(D^m) = \sum p_i^{-1}(D^{m_i})$$

and moreover,

$$d^*(\mathcal{I}_{D^m}) = \bigotimes {}_B\ p_i^*(\mathcal{I}_{D^{m_i}})$$

where we use $p_i$ generically to denote an $i$th coordinate projection. Therefore,

$$A(\mathcal{I}_{D^m}) \simeq \bigotimes {}_B\ p_i^* A(\mathcal{I}_{D^{m_i}})$$

as graded algebras, compatibly with the isomorphism

$$\mathcal{O}_{\prod\limits_{i=1}^{k} {}_B\ \mathrm{Sym}^{m_i}(X/B)} \simeq \bigotimes_{i=1}^{k} {}_B\ \mathcal{O}_{\mathrm{Sym}^{m_i}(X/B)}.$$

Now it is a general fact that Proj is compatible with tensor product of graded algebras, in the sense that

$$\mathrm{Proj}(\bigotimes {}_B\ A_i) \simeq \prod {}_B\ \mathrm{Proj}(A_i).$$

Consequently (1.2.2) induces another cartesian diagram with unramified vertical arrows

$$\begin{array}{ccc}
(\prod\limits_{i=1}^{k}{}_B\ X_B^{[m_i]})_{(Z.)} & \xrightarrow{\prod c'_{m_i}} & (\prod\limits_{i=1}^{k}{}_B\ B_{D^{m_i}}X_B^{(m_i)})_{(Z.)} \\
\uparrow & \square & \uparrow \\
H' & \to & S' \\
\downarrow & \square & \downarrow \\
(X_B^{[m]})_{(Z)} & \xrightarrow{c'_m} & (B_{D^m}X_B^{(m)})_{(Z)}.
\end{array}$$

Here $B_u v_{(Z)}$ in a blowup means, $\forall u, v$, the inverse image of $v_{(Z)}$ via the blowing-up map. To prove $c'_m$ is an isomorphism, it will suffice to prove that each of its fibres over a point $\gamma$ lying over $Z$ is schematically a point. Given $\gamma$, there is a unique point of $S'$ that maps to it and that on the other side maps to a point, say $\delta \in (\prod\limits_{i=1}^{k}{}_B\ B_{D^{m_i}}X_B^{(m_i)})_{(Z.)}$, that lies over $(Z.)$. Then $(\mathfrak{c}'_m)^{-1}(\gamma)$ corresponds isomorphically to $(\prod \mathfrak{c}'_{m_i})^{-1}(\delta)$. Therefore, it suffices to prove that $c'_{m_i}$ is an isomorphism for each $i$. The upshot of all this is that it suffices to prove $c = \mathfrak{c}_m$ is equivalent to the blow-up of $X_B^{(m)}$ in $D^m$, locally over a neighborhood of a cycle of the form $mp, p \in X$, and we may obviously assume that $p$ is a singular point of $\pi$.

1.3. **A local model.** We now reach the heart of the matter: an explicit construction, locally over the symmetric product, of the relative Hilbert scheme in terms of coordinates. This construction will stand us in good stead for the remainder of the paper, much beyond the proof of the Blowup Theorem. We begin with some preliminaries.

Fixing a fibre node $p$ as above, lying on a singular fibre $X_0$, an affine (if $p$ is a separating node) or analytic or formal (in any case) neighborhood $U$ of $p$ in $X$ so that $\pi$ is given on $U$ by pulling back a universal deformation

$$t = xy. \tag{1.3.1}$$

Since both the relative Hilbert scheme and the blowing-up process are compatible with pullback, we may as well assume that $U/B$ is itself given by (1.3.1). Then the relative cartesian product $X_B^m$, as subscheme of $X^m \times B$, is given locally by

$$x_1y_1 = ... = x_my_m = t.$$

Let $\sigma_i^x, \sigma_i^y, i = 0, ..., m$ denote the elementary symmetric functions in $x_1, ..., x_m$ and in $y_1, ..., y_m$, respectively, where we set $\sigma_0 = 1$. We note that these functions satisfy the relations

$$\sigma_m^y\sigma_j^x = t^j\sigma_{m-j}^y, \quad \sigma_m^x\sigma_j^y = t^j\sigma_{m-j}^x, \tag{1.3.2}$$

$$t^{m-i}\sigma_{m-j}^y = t^{m-i-j}\sigma_j^x\sigma_m^y, \quad t^{m-i}\sigma_{m-j}^x = t^{m-i-j}\sigma_j^y\sigma_m^x, \tag{1.3.3}$$

Putting the sigma functions together with the projection to $B$, we get a map

$$\sigma : \text{Sym}^m(U/B) \to \mathbb{A}^{2m}_B = \mathbb{A}^{2m} \times B$$
$$\sigma = ((-1)^m \sigma^x_m, ..., -\sigma^x_1, (-1)^m \sigma^y_m, ..., -\sigma^y_1, \pi^{(m)})$$

where $\pi^{(m)} : X^{(m)}_B \to B$ is the structure map.

**Lemma 1.3.** *$\sigma$ is an embedding locally near $mp$.*

*Proof.* It suffices to prove this formally, i.e. to show that $\sigma^x_i, \sigma^y_j, i, j = 1, ..., m$ generate topologically the completion $\hat{\mathfrak{m}}$ of the maximal ideal of $mp$ in $X^{(m)}_B$. To this end it suffices to show that any $\mathfrak{S}_m$-invariant polynomial in the $x_i, y_j$ is a polynomial in the $\sigma^x_i, \sigma^y_j$ and $t$. Let us denote by $R$ the averaging or symmetrization operator with respect to the permutation action of $\mathfrak{S}_m$, i.e.

$$R(f) = \frac{1}{m!} \sum_{g \in \mathfrak{S}_m} g^*(f).$$

Then it suffices to show that the elements $R(x^I y^J)$, where $x^I$ (resp. $y^J$) range over all monomials in $x_1, ..., x_m$ (resp. $y_1, ..., y_m$) are polynomials in the $\sigma^x_i, \sigma^y_j$ and $t$. Now the relations (1.3.2-1.3.3) on the image of $X^m_B$ easily implies that

$$R(x^I y^J) - R(x^I)R(y^J) = tF$$

where $F$ is an $\mathfrak{S}_m$-invariant polynomial in the $x_i, y_j$ of bidegree $(|I| - 1, |J| - 1)$, hence a linear combination of elements of the form $R(x^{I'} y^{J'}), |I'| = |I| - 1, |J'| = |J| - 1$. By induction, $F$ is a polynomial in the $\sigma^x_i, \sigma^y_j$ and clearly so is $R(x^I)R(y^J)$. Hence so is $R(x^I y^J)$ and we are done. □

*Remark* 1.4. It will follow from Theorem 1 and its proof that the equations (1.3.2-1.3.3) actually define the image of $\sigma$ scheme-theoretically (see Cor. 1.15 below); we won't need this, however.

Now we present a construction of our local model $\tilde{H}$. This is motivated by our study in [13] of the relative Hilbert scheme of a node. Let $C_1, ..., C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates $u_i, v_i$ on the $i$-th copy. Let

$$\tilde{C} \subset C_1 \times ... \times C_{m-1} \times B$$

be the subscheme defined by

(1.3.4) $$v_1 u_2 = t u_1 v_2, ..., v_{m-2} u_{m-1} = t u_{m-2} v_{m-1}.$$

Thus $\tilde{C}$ is a reduced complete intersection of divisors of type $(1, 1, 0, ..., 0), (0, 1, 1, 0, ..., 0)$ ,..., $(0, ..., 0, 1, 1)$ and it is easy to check that the fibre of $\tilde{C}$ over $0 \in B$ is

(1.3.5) $$\tilde{C}_0 = \bigcup_{i=1}^{m-1} \tilde{C}_i,$$

where

$$\tilde{C}_i = [1,0] \times ... \times [1,0] \times C_i \times [0,1] \times ... \times [0,1]$$

and that in a neighborhood of the special fibre $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over $B$. We may embed $\tilde{C}$ in $\mathbb{P}^{m-1} \times B$, relatively over $B$ using the plurihomogenous monomials

(1.3.6) $$Z_i = u_1 \cdots u_{i-1} v_i \cdots v_{m-1}, i = 1, ..., m.$$

These satisfy the relations

(1.3.7) $$Z_i Z_j = t^{j-i-1} Z_{i+1} Z_{j-1}, i < j-1$$

so they embed $\tilde{C}$ as a family of rational normal curves $\tilde{C}_t \subset \mathbb{P}^{m-1}, t \neq 0$ specializing to $\tilde{C}_0$, which is embedded as a nondegenerate, connected chain of $m-1$ lines.

Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, ..., a_{m-1}, d_0, ..., d_{m-1}$ and let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

(1.3.8) $$a_0 u_1 = t v_1, d_0 v_{m-1} = t u_{m-1}$$
$$a_1 u_1 = d_{m-1} v_1, ..., a_{m-1} u_{m-1} = d_1 v_{m-1}.$$

Set $L_i = p^*_{C_i} \mathcal{O}(1)$. Then consider the subscheme of $Y = \tilde{H} \times_B U$ defined by the equations

(1.3.9) $$F_0 := x^m + a_{m-1} x^{m-1} + ... + a_1 x + a_0 \in \Gamma(Y, \mathcal{O}_Y)$$

(1.3.10) $$F_1 := u_1 x^{m-1} + u_1 a_{m-1} x^{m-2} + ... + u_1 a_2 x + u_1 a_1 + v_1 y \in \Gamma(Y, L_1)$$

$$...$$

$$F_i := u_i x^{m-i} + u_i a_{m-1} x^{m-i-1} + ... + u_i a_{i+1} x + u_i a_i + v_i d_{m-i+1} y + ... + v_i d_{m-1} y^{i-1} + v_i y^i$$

(1.3.11) $$\in \Gamma(Y, L_i)$$

$$...$$

(1.3.12) $$F_m := d_0 + d_1 y_1 + ... + d_{m-1} y^{m-1} + y^m \in \Gamma(Y, \mathcal{O}_Y).$$

The following statement summarizes results from [13].

**Theorem 1.5.** (i) *$\tilde{H}$ is smooth and irreducible.*

(ii) *The ideal sheaf $\mathcal{I}$ generated by $F_0, ..., F_m$ defines a subscheme of $\tilde{H} \times_B X$ that is flat of length $m$ over $\tilde{H}$*

(iii) *The classifying map*

$$\Phi = \Phi_{\mathcal{I}} : \tilde{H} \to \textit{Hilb}_m(U/B)$$

*is an isomorphism.*

(iv) *$\Phi$ induces an isomorphism*

$$(\tilde{C})_0 = p^{-1}_{\mathbb{A}^{2m}}(0) \to \textit{Hilb}^0_m(X_0) = \bigcup_{i=1}^m C_i^m$$

*(cf. [13]) of the fibre of $\tilde{H}$ over $0 \in \mathbb{A}^{2m}$ with the punctual Hilbert scheme of the special fibre $X_0$, in such a way that the point $[u, v] \in \tilde{C}_i \sim C_i \sim \mathbb{P}^1$ corresponds to*

- *the subscheme $I_i^m(u/v) = (x^{m-i} + (u/v)y^i) \in C_i^m \subset$ Hilb$_m^0(X_0)$ if $uv \neq 0$,*
- *the subscheme $(x^{m+1-i}, y^i) \in C_i^m$ if $[u, v] = [0, 1]$,*
- *the subscheme $(x^{m-i}, y^{i+1}) \in C_i^m$ if $[u, v] = [1, 0]$.*

(v) *over $U_i$, a co-basis for the universal ideal $\mathcal{I}$ (i.e. a basis for $\mathcal{O}/\mathcal{I}$) is given by*

$$1, ..., x^{m-i}, y, ..., y^{i-1}.$$

(vi) *$\Phi$ induces an isomorphism of the special fibre $\tilde{H}_0$ of $H$ over $B$ with Hilb$_m(X_0)$, and $\tilde{H}_0 \subset \tilde{H}$ is a divisor with global normal crossings $\bigcup_{i=0}^{m} D_i^m$ where each $D_i^m$ is smooth, birational to $(x-$ axis$)^{m-i} \times (y-$ axis$)^i$, and has special fibre $C_i^m$ under the cycle map $p_{\mathbb{A}^m}$.*

*Proof.* The smoothness of $\tilde{H}$ is clear from the defining equations equations and also follows from smoothness of $\text{Hilb}_m(U/B)$ once (ii) and (iii) are proven. To that end consider the point $q_i, i = 1, ..., m$, on the special fibre of $\tilde{H}$ over $\mathbb{A}_B^{2m}$ with coordinates

$$v_j = 0, \ \forall j < i; u_j = 0, \ \forall j \geq i.$$

Then $q_i$ has an affine neighborhood $U_i$ in $\tilde{H}$ defined by

$$U_i = \{u_j = 1, \ \forall j < i; \ v_j = 1, \ \forall j \geq i\},$$

and these $U_i, i = 1, ..., m$ cover a neighborhood of the special fibre of $\tilde{H}$. Now the generators $F_i$ admit the following relations:

$$u_{i-1}F_j = u_j x^{i-1-j} F_{i-1}, \ 0 \leq j < i-1; \ v_i F_j = v_j y^{j-i} F_i, \ m \geq j > i$$

where we set $u_i = v_i = 1$ for $i = 0, m$. Hence $\mathcal{I}$ is generated there by $F_{i-1}, F_i$ and assertions (ii), (iii) follow directly from Theorems 1,2 and 3 of [13] and (iv) is obvious.

As for (v), it follows immediately from the definition of the $F_i$, plus the fact just noted that, over $U_i$, the ideal $\mathcal{I}$ is generated by $F_{i-1}, F_i$, and that on $U_i$, we have $u_{i-1} = v_i = 1$. Finally (vi) is contained in [13], Thm. 2.

□

1.4. **Excursions about $H_m$.** In light of Theorem 1.5, we identify a neighborhood $H_m$ of the special fibre in $\tilde{H}$ with a neighborhood of the punctual Hilbert scheme (i.e. $\mathfrak{c}_m^{-1}(mp)$) in $X_B^{[m]}$, and note that the projection $H_m \to \mathbb{A}^{2m} \times B$ coincides generically, hence everywhere, with $\sigma \circ \mathfrak{c}_m$. Hence $H_m$ may be viewed as the subscheme

of $\text{Sym}^m(U/B) \times_B \tilde{C}$ defined by the equations

$$\begin{aligned} \sigma_m^x u_1 &= t v_1, \\ \sigma_{m-1}^x u_1 = \sigma_1^y v_1, ..., \sigma_1^x u_{m-1} &= \sigma_{m-1}^y v_{m-1}, \\ t u_{m-1} &= \sigma_m^y v_{m-1} \end{aligned} \tag{1.4.13}$$

Alternatively, in terms of the $Z$ coordinates, $H_m$ may be defined as the subscheme of $\text{Sym}^m(U/B) \times \mathbb{P}_Z^{m-1} \times B$ defined by the relations (1.3.7), which define $\tilde{C}$, together with

$$\sigma_i^y Z_i = \sigma_{m-i}^x Z_{i+1}, \ i = 1, ..., m-1 \tag{1.4.14}$$

Now having determined the structure of $c_m$ along its 'most special' fibre $c_m^{-1}(m(0,0))$, we can easily, and usefully, determine its structure along other fibres, as follows. For simplicity we assume for the rest of this subsection that $B$ is a smooth curve, with local coordinate $t$, and that the singular fibre $X_0$ has a unique node $p$, with $U$ being a neighborhood of $p$ in $X$.

Let $X', X"$ denote the $x, y$ axes, respectively in $U_0 = X_0 \cap U$, with their respective origins $0', 0"$. If the special fibre $X_0$ is reducible, then $X', X"$ globalize to the two components of the normalization (which will be denoted in the same way if no undue confusion results). If $X_0$ is irreducible, then both $X'$ and $X"$ globalize to the normalization. For any pair of natural numbers $(a, b), 0 < a + b < m$, set

$$X^{(a,b)} = X'^{(a)} \times X"^{(b)}$$

(which globalizes to a component (the unique one, if $X_0$ is irreducible) of the normalization of $X_0^{a+b}$. Then we have a natural map

$$X^{(a,b)} \to \text{Sym}^m(U_0) \subset \text{Sym}^m(U/B)$$

given by

$$(\sum m_i x_i, \sum n_j y_j) \mapsto \sum m_i(x_i, 0) + \sum n_j(0, y_j) + (m - a - b)(0, 0).$$

This map is clearly birational to its image, which we denote by $\bar{X}^{(a,b)}$. Thus $X^{(a,b)}$ coincides with the normailzation of $\bar{X}^{(a,b)}$. It is clear that $\bar{X}^{(a,b)}$ is defined by the equations

$$\sigma_m^x = ... = \sigma_{a+1}^x = 0, \sigma_m^y = ... = \sigma_{b+1}^y = 0.$$

A point

$$c \in \bar{X}^{(a,b)} - (\bar{X}^{(a+1,b)} \cup \bar{X}^{(a,b+1)}),$$

i.e. a cycle in which $(0, 0)$ appears with multiplicity exactly $n = m - a - b$, is said to be of *type* $(a, b)$. Type yields a natural stratification of the symmetric product $X_0^{(m)}$. Now let $H^{(a,b)}$ be the closure of the locus of schemes whose cycle if of type $(a, b)$. i.e.

$$H^{(a,b)} = \text{closure}(c_m^{-1}(\bar{X}^{(a,b)} - (\bar{X}^{(a+1,b)} \cup \bar{X}^{(a,b+1)}))) \subset H_m \tag{1.4.15}$$

Clearly the restriction of $c_m$ on $H^{(a,b)}$ factors through a map

$$\tilde{c}_m : H^{(a,b)} \to X^{(a,b)},$$
$$\tilde{c}_m = ((\sigma_1^x, ..., \sigma_a^x), (\sigma_1^y, ..., \sigma_b^y))$$

Approaching the 'origin cycle' $m(0,0)$ through cycles of type $(a,b)$, i.e. approaching the point $(a0', b0")$ on $X^{(a,b)}$, means that $a$ (resp. $b$) points are approaching the origin $0'$ (resp. $0"$) along the $x$ (resp. $y$)-axis. For a cycle $c$ of type $(a,b)$, we have, for all $j \leq b$, that $\sigma_j^y \neq 0, \sigma_{m-j}^x = 0$, hence by the equations (1.3.8) (setting each $a_i = \sigma_{m-i}^x, d_i = \sigma_{m-i}^y$), we conclude $v_j = 0$; thus

$$v_1 = ... = v_b = 0; \tag{1.4.16}$$

similarly, for all $j \leq a$, we have $\sigma_{m-j}^y = 0, \sigma_j^x \neq 0$, hence again by the equations (1.3.8) , we conclude $u_{m-j} = 0$; thus

$$u_{m-1} = ... = u_{m-a} = 0. \tag{1.4.17}$$

Consequently, the fibre of $\mathfrak{c}_m$ over this point is schematically

$$c_m^{-1}(c) = \tilde{c}_m^{-1}(c) = \bigcup_{i=b+1}^{m-a-1} C_i^m, \tag{1.4.18}$$

provided $a + b \leq m - 2$. If $a + b = m - 1$, the fibre is the unique point given by

$$v_1 = ... = v_b = u_{b+1} = ... = u_{m-1} = 0$$

(this is the point denoted $Q_{b+1}^m$ in [13], i.e. the subscheme with ideal $(x^{m-b}, y^{b+1})$). As $c$ approaches the 'origin' $(a0', b0")$ in $X^{(a,b)}$, the equations (1.4.16),(1.4.17) persist, so we conclude

$$\tilde{c}_m^{-1}((a0', b0")) = \begin{cases} \bigcup_{i=b+1}^{m-a-1} C_i^m, a + b \leq m - 2, \\ Q_{b+1}^m, a + b = m - 1. \end{cases} \tag{1.4.19}$$

Thus, working in $H^{(a,b)}$ over $X^{(a,b)}$, the special fibre is the same as the general fibre. Moreover as subscheme of $H_m \times_{X^{(m)}} X^{(a,b)}$, $H^{(a,b)}$ is defined by the equations (1.4.17) and (1.4.16). And in the special case $a + b = m - 2$, we see that $H^{(a,b)}$ forms a $\mathbb{P}^1$-bundle with fibre $C_{b+1}^m$, locally near $(a0', b0")$. This is a so-called *node scroll*, to be discussed further below.

Incidentally, in case $a + b = m$, a similar but simpler analysis shows that the fibre $\tilde{c}_m^{-1}((a0', b0"))$ coincides with $C_b^m \simeq \mathbb{P}^1$ if $1 \leq b \leq m-1$ and with the single point $Q_{b+1}^m$ if $b = 0, m$. This, of course, is contained in part (vi) of Theorem 1.5 above.

Summarizing this discussion, a bit more usefully, in terms of $Z$ coordinates, we have

**Corollary 1.6.** *For any $\ell_1 + \ell_2 \leq m, \ell_1, \ell_2 \geq 0$, we have, in any component of the locus $c_m^{-1}((X')^{(\ell_1)} \times (X")^{(\ell_2)}) \subset H_m$ dominating $(X')^{(\ell_1)} \times (X")^{(\ell_2)}$, that*

$$Z_i = 0, \forall i \geq \max(m - \ell_1 + 1, \ell_2 + 2), \quad \forall i \leq \min(\ell_2, m - \ell_1 - 1). \tag{1.4.20}$$

*In particular, if $\ell_1 + \ell_2 = m$ (resp. $\ell_1 + \ell_2 < m$), the only nonzero $Z_i$ are where*

$$i \in [\ell_2, \ell_2 + 1] \cap [1, m] \ \ (\textit{resp.}\ \ i \in [l_2 + 1, m - \ell_1] \cap [1, m]).$$

Taking to account the linear relations (1.4.14), we also conclude

**Corollary 1.7.** (i) *For any component $(X')^\ell \times (X")^{m-\ell}, 0 < \ell < m$ of the special fibre of $U_B^{(m)}$, the unique dominant component of $c_m^{-1}((X')^\ell \times (X")^{m-\ell})$ coincides with the graph of rational map*

$$(X')^\ell \times (X")^{m-\ell} \dashrightarrow \mathbb{P}^1_{Z_\ell, Z_{\ell+1}} \subset \mathbb{P}_Z^{m-1} \tag{1.4.21}$$

*defined by*

$$[\sigma_\ell^x, \sigma_{m-\ell}^y];$$

(ii) *ditto for $\ell = m$ (resp. $\ell = 0$), with the constant rational map to $[1, 0, ...]$ (resp. $[..., 0, 1]$);*

(iii) *ditto over $(X')^\ell \times (X")^{m-\ell-1}, 0 \leq \ell \leq m-1$, with the constant map to $[..., 0, 1_{m-\ell}, 0...]$.*

Now on the other hand, working near a cycle $c$ of type $(a, b)$ and fixing its off-node portion, say of length $k = m - n$, we also have an obvious identification of the same (general) fibre of $H^{(a,b)}/X^{(a,b)}$ over $c$ as the special fibre in a local model $H_n$ for the length-$n$ Hilbert scheme. Namely, if we let $c' = n(0, 0)$ be the part of $c$ supported at the origin, then essentially the same fibre $c_m^{-1}(c)$ can also be written as

$$c_n^{-1}(c') = \bigcup_{j=1}^{n} C_j^n \tag{1.4.22}$$

and naturally $C_j^n$ corresponds to $C_{j+b}^m = C_{j+b}^{n+a+b}$. Of course under the identification of Theorem 1.5, $c_n^{-1}(c')$ corresponds to the punctual Hilbert scheme $Hilb_n^0(X_0)$. So we conclude that the $j$-th punctual length-$n$ Hilbert scheme component at $c$ specializes to the $(j + b)$-th length-$m$ Hilbert scheme component at $m(0, 0)$ as $c$ specializes to $m(0, 0)$ over the normalization $X^{(a,b)}$. Note that the analogous fact holds for any cycle of multiplicity $n$ at $(0, 0)$ specializing to one of multiplicity $m$ at $(0, 0)$, even if its total degree is higher. Thus we have

**Lemma 1.8.** *As a cycle $c$ on $X_0$, having multiplicity $n$ at the origin, approaches a cycle $d$ with multiplicity $m > n$ at the origin, so that for some $a, b$ with $a + b = m - n$, $a$ points approach along the $x$-axis and $b$ points along the $y$-axis, the punctual Hilbert scheme component $C_j^n$ over $c$ specializes smoothly to $C_{j+b}^m$ at $d$.*

We also see, comparing (1.4.18) and (1.4.19), that the fibre $\tilde{c}_m^{-1}(c)$ is 'constant', i.e. it doesn't depend on $c$ as it moves in $X^{(a,b)}$. Moreover, as $c$ moves in $X^{(a,b)}$, the individual components of this fibre, which have to do with branches of $X_0^{(m-a-b)}$ at $(m-a-b)(0,0)$, $X_0$ being the entire singular fibre (or what is the same, branches of $X_0^m$ generically along $X^{(a,b)}$), remain well defined (i.e. not interchanged by monodromy), and specialize smoothly to similar components on lower-dimensional strata. Note that this is true even if $X_0$ is (globally) irreducible, in which case the other $a+b$ branches of $X_0$, $a$ from $X'$ and $b$ from $X$" are globally interchangeable. Therefore:

**Lemma 1.9.** *Notation as in (1.4.15)* et seq.*, we have*

(1.4.23)
$$H^{(a,b)} = \bigcup_{j=1}^{n-1} F_j^{(a|b)}$$

*where $F_j^{(a,b)} \subset H_m$ is the subscheme defined by*

(1.4.24)
$$v_1 = ... = v_{j+b} = u_{j+b+1} = .... = u_{m-1} = 0$$

*and*

$$\tilde{c}_m : F_j^{(a|b)} \to X^{(a,b)}$$

*is a $\mathbb{P}^1$ bundle with general fibre $C_j^n$. Moreover, $\tilde{c}_m|_{F(j:a|b)}$ admits two disjoint sections with respective general fibres the points corresponding to the punctual schemes of type $Q_j^n, Q_{j+1}^n$.*

Fixing $a, b$ for now, the $F_j = F_j^{(a|b)}$ are special (but typical) cases of what are called *node scrolls*. It follows from the lemma that we can write

$$F_j = \mathbb{P}(L_j^n \oplus L_{j+1}^n)$$

for certain line bundles $L_j^n$ on $X^{(a,b)}$, corresponding to the disjoint sections $Q_j^n, Q_{j+1}^n$, where the difference $L_j^n - L_{j+1}^n$ is uniquely determined (we use additive notation for the tensor product of line bundles and quotient convention for projective bundles). The identification of a natural choice for both these line bundles, using methods to be developed later in this section, will be taken up in the next section and plays an important role in the enumerative geometry of the Hilbert scheme. But the difference $L_j^n - L_{j+1}^n$, and hence the intrinsic structure of the node scroll $F_j$, may already be computed now, as follows.

Write

$$Q_j = \mathbb{P}(L_j), Q_{j+1} = \mathbb{P}(L_{j+1})$$

for the two special sections of type $Q_j^n, Q_{j+1}^n$ respectively. Let

$$D_{0'}, D_{0"} \subset X^{(a,b)}$$

be the divisors comprised of cycles containing $0'$ (resp. $0$”). In the local model, these are given locally by the respective equations

$$D_{0'} = (\sigma^x_a), D_{0"} = (\sigma^y_b).$$

**Lemma 1.10.** *We have, using the quotient convention for projective bundles,*

$$F_j = \mathbb{P}_{X^{(a,b)}}(\mathcal{O}(-D_{0'}) \oplus \mathcal{O}(-D_{0"})), j = 1, ..., n-1. \tag{1.4.25}$$

*Proof.* Our key tool is a $\mathbb{C}^*$- parametrized family of sections ’interpolating’ between $Q_j$ and $Q_{j+1}$. Namely, note that for any $s \in \mathbb{C}^*$, there is a well-defined section $I_s$ of $F_j$ whose fibre over a general point $z \in X^{(a,b)}$ is the scheme

$$I_s(z) = (sx^{n-j} + y^j) \coprod \mathrm{sch}(z),$$

where $\mathrm{sch}(z)$ is the unique subscheme of length $a+b$, disjoint from the nodes, corresponding to $z$.

*Claim:* The fibre of $I_s$ over a point $z \in D_{0'}$ (resp. $z \in D_{0"}$) is $Q^n_j$ (resp. $Q^n_{j+1}$).

*Proof of claim.* Indeed set-theoretically the claim is clear from the fact the this fibre corresponds to a length-$n$ punctual scheme meeting the $x$-axis (resp. $y$-axis) with multiplicity at least $n-j+1$ (resp. $j+1$).

To see the same thing schematically, via equations in the local model $H_{n+1}$, we proceed as follows. Working near a generic point $z_0 \in D_{0'}$ we can, discarding distal factors supported away from the nodes, write the singleton scheme corresponding nearby cycle $z$ as $\mathrm{sch}(z) = (x-c, y)$ where $c \to 0$ as $z \to z_0$, and then

$$I_s(z) = (sx^{n-j} + y^j)(x-c, y) = (sx^{n-j+1} - csx^{n-j} - cy^j, y^{j+1}).$$

Thus, in terms of the system of generators (1.3.9), $I_s(z)$ is defined locally by

$$cu_j - sv_j = 0 \tag{1.4.26}$$

(with other $[u_k, v_k]$ coordinates either $[1,0]$ for $k < j$ or $[0,1]$ for $k > j$. The limit of this as $c \to 0$ is $[u_j, v_j] = [1, 0]$, which is the point $Q_j$. *QED Claim.*

Clearly $I_s$ doesn’t meet $Q_j$ or $Q_{j+1}$ away from $D_{0'} \cup D_{0"}$. Therefore, we have

$$I_s \cap Q_j = Q_j.D_{0'}, \tag{1.4.27}$$

$$I_s \cap Q_{j+1} = Q_{j+1}.D_{0"}; \tag{1.4.28}$$

an easy calculation in the local model shows that the intersection is transverse. Because $Q_j \cap Q_{j+1} = \emptyset$, it follows that

$$I_a \sim Q_j + D_{0'}.F_j \tag{1.4.29}$$

$$\sim Q_{j+1} + D_{0"}.F_j. \tag{1.4.30}$$

These relations also follow from the fact, which comes simply from setting $s = 0$ or dividing by $s$ and setting $s = \infty$ in (1.4.26), that

$$\lim_{s\to 0} I_s = Q_j + D_{0'}.F_j, \lim_{s\to\infty} I_s = Q_{j+1} + D_{0"}.F_j \tag{1.4.31}$$

It then follows that

$$(Q_j)^2 = Q_j.(I_s - D_{0'}.F_j) = Q_j.(Q_{j+1} + (D_{0"} - D_{0'}).F_j),$$

hence

$$(Q_j)^2 = Q_j(D_{0"} - D_{0'}), \tag{1.4.32}$$

therefore finally

$$L_j^n - L_{j+1}^n = D_{0"} - D_{0'}. \tag{1.4.33}$$

This proves the Lemma. □

1.5. **Globalization.** We now wish to extend the discussion of the last subsection, in paticular the notion of node scrolls, to the general case, with higher-dimensional base and fibres with more than one node: in this case a node scroll becomes a $\mathbb{P}^1$-bundle over a relative Hilbert scheme associated to a 'boundary family' of $X/B$, i.e a family obtained essentially as the partial normalization of the subfaily of $X/B$ lying over the normalization of a component of the locus of singular curves in $B$ (a.k.a. the boundary of $B$).

To this end let $\pi : X \to B$ now denote an arbitrary flat family of nodal curves of arithmetic genus $g$ over an irreducible base, with smooth generic fibre. In order to specify the additional information required to define a node scroll, we make the following definition.

**Definition 1.11.** *A boundary datum for $X/B$ consists of*

- (i) *an irreducible variety $T$ with a map $\delta : T \to B$ unramified to its image;*
- (ii) *a lifting $\theta : T \to X$ of $\delta$ such that each $\theta(t)$ is a node of $X_{\delta(t)}$;*
- (iii) *a labelling, continuous in $t$, of the two branches of $X_{\delta(t)}$ along $\theta(t)$ as $x$-axis and $y$-axis.*

*Given such a datum, the associated boundary family $X_T^\theta$ is the normalization (= blowup) of the base-changed family $X \times_B T$ along the section $\theta$, i.e.*

$$X_T^\theta = \mathrm{B}\ell_\theta(X \times_B T),$$

*viewed as a family of curves of genus $g-1$ with two, everywhere distinct, individually defined marked points $\theta_x, \theta_y$. We denote by $\phi$ the natural map fitting in the diagram*

$$\begin{array}{ccc} X_T^\theta & & \\ \downarrow & \overset{\phi}{\searrow} & \\ X \times_B T & \to & X \\ \downarrow & & \downarrow \\ T & \overset{\delta}{\to} & B. \end{array}$$

Note that any component $T_0$ of the boundary of $B$, i.e. the (divisorial) locus of singular fibres, gives rise to (finitely many) boundary data in this sense: first consider a component $T_1$ of the normalization of $T_0 \times_B \operatorname{sing}(X/B)$, which already admits a node-valued lifting $\theta_1$ to $X$, then further base-change by the normal cone of $\theta_1(T_1)$ in $X$ (which is 2:1 unramified, possibly disconnected, over $T_1$), to obtain a boundary datum as above. 'Typically', the curve corresponding to a general point in $T_0$ will have a single node $\theta$ and then the degree of $\delta$ will be 1 or 2 depending on whether the branches along $\theta$ are distinguishable in $X$ or not (they always are distinguishable if $\theta$ is a separating node and the separated subcurves have different genera). Proceeding in this way and taking all components which arise, we obtain finitely many boundary data which 'cover', in an obvious sense, the entire boundary of $B$. Such a collection, weighted so that each boundary component $T_0$ has total weight $= 1$ is called a *covering system of boundary data.*

**Proposition-definition 1.12.** *Given a boundary datum $(T, \delta, \theta)$ for $X/B$ and natural numbers $1 \leq j < n$, there exists a $\mathbb{P}^1$-bundle $F^n_j(\theta)$, called a node scroll over the Hilbert scheme $(X^\theta_T)^{[m-n]}$, endowed with two disjoint sections $Q^n_{j,j}, Q^n_{j+1,j}$, together with a surjective map generically of degree equal to $\deg(\delta)$ of*

$$\bigcup_{j=1}^{n-1} F^n_j(\theta) := \coprod_{j=1}^{n-1} F^n_j(\theta) / \coprod_{j=1}^{n-2} (Q^n_{j+1,j} \sim Q^n_{j+1,j+1})$$

*onto the closure in $X^{[m]}_B$ of the locus of schemes having length precisely $n$ at $\theta$, so that a general fibre of $F^n_j(\theta)$ corresponds to the family $C^n_j$ of length-$n$ schemes at $\theta$ generically of type $I^n_j(a)$, with the two nonprincipal schemes $Q^n_j, Q^n_{j+1}$ corresponding to $Q^n_{j,j}, Q^n_{j,j+1}$ respectively. We denote by $\delta^n_j$ the natural map of $F^n_j(\theta)$ to $X^{[m]}_B$.*

*Proof-construction.* The scroll $F^n_j(\theta)$ is defined as follows. Fixing the boundary data, consider first the locus

$$\bar{F}^n_j \subset T \times_B X^{[m]}_B$$

consisting of compatible pairs $(t, z)$ such that $z$ is in the closure of the set of schemes which are of type $I^n_j$ (i.e. $x^{n-j} + ay^j, a \in \mathbb{C}^*$) at $\theta(t)$, with respect to the branch order $(\theta_x, \theta_y)$. The discussion of the previous subsection shows that the general fibre of $\bar{F}_j$ under the cycle map is a $\mathbb{P}^1$, namely a copy of $C^n_j$; moreover the closure of the locus of schemes having multiplicity $n$ at $\theta$ is the union $\bigcup\limits_{j-1}^{n-1} \bar{F}^n_j$.

In fact locally over a cycle having multiplicity precisely $n+e$ at $\theta$, $\bar{F}^n_j$ is a union of components $\bar{F}^{(n:a,b)}_j, a+b=e$, where $\bar{F}^{(n:a,b)}_j$ maps to $(X')^a \times (X")^b$ and is defined in the local model $H_{n+e}$ by is defined by the vanishing of all $Z_i, i \neq j+b, j+b+1$ or alternatively, in terms of $u, v$ coordinates, by

$$v_1 = ... = v_{j+b} = u_{j+b+1} = ... = u_{n+e} = 0$$

Then $F_j^n(\theta)$ is the locus

$$\{(w,t,z)\in (X_T^\theta)^{[m-n]}\times_T \bar{F}_j^n : \phi_*(c_{m-n}(w))+n\theta=c_m(z),\} \tag{1.5.1}$$

where $\phi : X^\theta \to X$ is the natural map, clutching together $\theta_x$ and $\theta_y$, and $\phi_*$ is the induced push-forward map on cycles. Then the results of the previous section show that $F_j^n(\theta)$ is locally defined near a cycle having multiplicity $b$ at $\theta_y$, e.g. by the vanishing of the $Z_i, i\neq j+b, j+b+1$ on

$$\{(w,u,Z)\in (X_T^\theta)^{[m-n]}\times X_B^{(e)}\times \mathbb{P}^{n+e} : \phi_*(c_{m-n}(w))_\theta+n\theta=u$$

where $._\theta$ indicates the portion near $\theta$. The latter locus certainly projects isomorphically to its image in $(X_T^\theta)^{[m-n]}\times\mathbb{P}^{n+e}$, hence $F_j^n(\theta$ is a $\mathbb{P}^1$-bundle over $(X_T^\theta)^{[m-n]}$. Since $F_j^n(\theta)$ admits the two sections $Q_{j,j}^n, Q_{j+1,j}^n$, it is the projectivization of a decomposable rank-2 vector bundle. $\square$

In addition to the node scroll $F_j^n(\theta)$, we will also consider its ordered version, i.e.

$$OF_j^n(\theta)=F_j^n(\theta)\times_{(X_T^\theta)^{(m-n)}}(X_T^\theta)^{m-n}, \tag{1.5.2}$$

and and similarly for $\bar{OF}_j^n(\theta)$. Also, for each $n$-tuple $I\subset[1,m]$, the corresponding locus in $X_B^{\lceil m\rceil}$, i.e.

$$OF_j^I=\{(w,t,z)\in (X_T^\theta)^{\lceil m-n\rceil}\times_T \bar{OF}_j^n : \phi_*(oc_{m-n}(w))+\sum_{i\in I}p_i^*(\theta)=c_m(z)\}, \tag{1.5.3}$$

this being the 'node scroll inserted over the $I$-indexed coordinates.

1.6. **Reverse engineering.** Our task now is effectively to 'reverse-engineer' an ideal in the $\sigma$'s whose syzygies are given by (1.4.14) and (1.3.7) . To this end, it is convenient to introduce order in the coordinates. Thus let $OH_m=H_m\times_{\mathrm{Sym}^m(U/B)} U_B^m$, so we have a cartesian diagram

$$\begin{array}{ccc} OH_m & \xrightarrow{\varpi_m} & H_m \\ o\mathfrak{c}_m\downarrow & \square & \downarrow\mathfrak{c}_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

and its global analogue

$$\begin{array}{ccc} X_B^{\lceil m\rceil} & \xrightarrow{\varpi_m} & X_B^{[m]} \\ o\mathfrak{c}_m\downarrow & \square & \downarrow\mathfrak{c}_m \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)} \end{array}$$

Note that $X_B^{(m)}$ is normal and Cohen-Macaulay: this follows from the fact that it is a quotient by $\mathfrak{S}_m$ of $X_B^m$, which is a locally complete intersection with singular locus of codimension $\geq 2$ (in fact, $>2$, since $X$ is smooth). Alternatively, normality of $X_B^{(m)}$ follows from the fact that $H_m$ is smooth and the fibres of

$\mathfrak{c}_m : H_m \to X_B^{(m)}$ are connected (being products of connected chains of rational curves). Note that $\omega_m$ is simply ramified generically over $D^m$ and we have

$$\omega_m^*(D^m) = 2OD^m$$

where

$$OD^m = \sum_{i<j} D^m_{i,j}$$

where $D^m_{i,j} = p^{-1}_{i,j}(OD^2)$ is the locus of points whose $i$th and $j$th components coincide. To prove $\mathfrak{c}_m$ is equivalent to the blowing-up of $D^m$ it will suffice to prove that $o\mathfrak{c}_m$ is equivalent to the blowing-up of $2OD^m = \omega_m^*(D^m)$ which in turn is equivalent to the blowing-up of $OD^m$. The advantage of working with $OD^m$ rather than its unordered analogue is that at least some of its equations are easy to write down: let

$$v_x^m = \prod_{1\leq i<j\leq m} (x_i - x_j),$$

and likewise for $v_y^m$. As is well known, $v_x^m$ is the determinant of the Van der Monde matrix

$$V_x^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-1} & \dots & x_m^{m-1} \end{bmatrix}.$$

Also set

$$\tilde{U}_i = \varpi_m^{-1}(U_i),$$

where $U_i$ is as in (1.3.7), being a neighborhood of $q_i$ on $H_m$. Then in $U_1$, the universal ideal $\mathcal{I}$ is defined by

$$F_0, \quad F_1 = y + (\text{function of } x)$$

and consequently the length-$m$ scheme corresponding to $\mathcal{I}$ maps isomorphically to its projection to the $x$-axis. Therefore over $\tilde{U}_1 = \varpi_m^{-1}(U_1)$, where $F_0$ splits as $\prod(x - x_i)$, the equation of $OD^m$ is simply given by

$$G_1 = v_x^m.$$

Similarly, the equation of $OD^m$ in $\tilde{U}_m$ is given by

$$G_m = v_y^m.$$

New let

$$\Xi : OH_m \to \mathbb{P}^{m-1}$$

be the morphism corresponding to $[Z_1, ..., Z_m]$, and set $L = \Xi^*(\mathcal{O}(1))$. Note that $\tilde{U}_i$ coincides with the open set where $Z_i \neq 0$, so $Z_i$ generates $L$ over $\tilde{U}_i$. Let

$$O\Gamma^{(m)} = o\mathfrak{c}_m^{-1}(OD^m).$$

This is a $1/2-$Cartier divisor because $2O\Gamma^{(m)} = \varpi_m^{-1}(\Gamma^{(m)})$ and $\Gamma^{(m)}$ is Cartier, $H_m$ being smooth. In any case, the ideal $\mathcal{O}(-O\Gamma^{(m)})$ is a divisorial sheaf (reflexive of rank 1). Our aim now is to construct an isomorphism

$$\gamma : \mathcal{O}(-O\Gamma^{(m)}) \to L. \tag{1.6.1}$$

As we shall see, this isomorphism will easily imply Theorem 1. To construct $\gamma$, it suffices to specify it on each $\tilde{U}_i$.

1.7. **Mixed Van der Mondes and conclusion of proof.** A clue as to how the latter might be done comes from the relations (1.4.2-1.4.3). Thus, set

$$G_i = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} v_x^m = \frac{(\sigma_m^y)^{i-1}}{t^{(i-1)(m-i/2)}} G_1, \quad i = 2, ..., m. \tag{1.7.1}$$

Thus,

$$G_2 = \frac{\sigma_m^y}{t^{m-1}} G_1, G_3 = \frac{\sigma_m^y}{t^{m-2}} G_2, ..., G_{i+1} = \frac{\sigma_m^y}{t^{m-i}} G_i, i = 1, ..., m-1. \tag{1.7.2}$$

An elementary calculation shows that if we denote by $V_i^m$ the 'mixed Van der Monde' matrix

$$V_i^m = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_m \\ \vdots & & \vdots \\ x_1^{m-i} & \dots & x_m^{m-i} \\ y_1 & \dots & y_m \\ \vdots & & \vdots \\ y_1^{i-1} & \dots & y_m^{i-1} \end{bmatrix}$$

then we have

$$G_i = \pm \det(V_i^m), i = 1, ..., m. \tag{1.7.3}$$

⌜ Indeed for $i = 1$ this is standard; for general $i$, it suffices to prove the analogue of (1.7.2) for the mixed Van der Monde determinants. For this, it suffices to multiply each $j$th column of $V_i^m$ by $y_j$, and factor a $t = x_j y_j$ out of each of rows $2, ..., m-i+1$, which yields

$$\sigma_m^y \det(V_i^m) = (-1)^{m-i+1} t^{m-i} V_{i+1}^m. \tag{1.7.4}$$

⌞ From (1.7.3) it follows, e.g., that $G_m$ as given in (1.7.1) coincides with $v_y^m$. Next we claim

**Lemma 1.13.** *$G_i$ generates $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_i$.*

*Proof of Lemma.* This is clearly true where $t \neq 0$ and it remains to check it along the special fibre $OH_{m,0}$ of $OH_m$ over $B$. Note that $OH_{m,0}$ is a sum of components of the form

$$\Theta_I = \text{Zeros}(x_i, i \notin I, y_i, i \in I), I \subseteq \{1, ..., m\}, \tag{1.7.5}$$

none of which is contained in the singular locus of $OH_m$. Set

$$\Theta_i = \bigcup_{|I|=i} \Theta_I.$$

Note that

$$\tilde{C}_i \times 0 \subset \Theta_i, i = 1, ..., m-1$$

and therefore

$$\tilde{U}_i \cap \Theta_j = \emptyset, j \neq i-1, i.$$

Note that $y_i$ vanishes to order 1 (resp. 0) on $\Theta_I$ whenever $i \in I$ (resp. $i \notin I$). Similarly, $x_i - x_j$ vanishes to order 1 (resp. 0) on $\Theta_I$ whenever both $i, j \in I^c$ (resp. not both $i, j \in I^c$). From this, an elementary calculation shows that the vanishing order of $G_j$ on every component $\Theta$ of $\Theta_k$ is

$$\text{ord}_\Theta(G_j) = (k-j)^2 + (k-j). \tag{1.7.6}$$

We may unambiguously denote this number by $\text{ord}_{\Theta_k}(G_j)$. Since this order is nonnegative for all $k, j$, it follows firstly that the rational function $G_j$ has no poles, hence is in fact regular on $X_B^m$ near $mp$ (recall that $X_B^m$ is normal); of course, regularity of $G_j$ is also immediate from (1.7.3). Secondly, since this order is zero for $k = j, j-1$, and $\Theta_j, \Theta_{j-1}$ contain all the components of $OH_{m,0}$ meeting $\tilde{U}_j$, it follows that in $\tilde{U}_j$, $G_j$ has no zeros besides $O\Gamma^{(m)} \cap \tilde{U}_j$, so $G_j$ is a generator of $\mathcal{O}(-O\Gamma^{(m)})$ over $\tilde{U}_j$. QED Lemma. □

The Lemma yields a set of generators for the ideal of $OD^m$:

**Corollary 1.14** (of Lemma). *The ideal of $OD^m$ is generated, locally near $p^m$, by $G_1, ..., G_m$.*

*Proof.* If $Q$ denotes the cokernel of the map $m\mathcal{O}_{X^m} \to \mathcal{O}_{X^m}(-OD^m)$ given by $G_1, ..., G_m$, then $c_m^*(Q) = 0$ by the Lemma, hence $Q = 0$, so the $G$'s generate $\mathcal{O}_{X^m}(-OD^m)$. □

Now we can construct the desired isomorphism $\gamma$ as in (1.6.1), as follows. Since $Z_j$ is a generator of $L$ over $\tilde{U}_j$, we can define our isomorphism $\gamma$ over $\tilde{U}_j$ simply by specifying that

$$\gamma(G_j) = Z_j \text{ on } \tilde{U}_j.$$

Now to check that these maps are compatible, it suffices to check that

$$G_j/G_k = Z_j/Z_k$$

as rational functions (in fact, units over $\tilde{U}_j \cap \tilde{U}_k$). But the ratios $Z_j/Z_k$ are determined by the relations (1.4.14), while $G_j/G_k$ can be computed from (1.7.2), and it is trivial to check that these agree.

Now we can easily complete the proof of Theorem 1. The existence of $\gamma$, together with the universal property of blowing up, yields a morphism

$$Bc_m : OH_m \to B_{OD^m} X_B^m$$

which is clearly proper and birational, hence surjective. On the other hand, the fact that the $G$'s generate the ideal of $OD^m$, and correspond to the $Z$ coordinates on $OH_m \subset X_B^m \times \mathbb{P}^{m-1}$, implies that $Bc_m$ is a closed immersion. Therefore $Bc_m$ is an isomorphism. □

**Corollary 1.15.** *The image of the relative symmetric product $X_B^{(m)}$ under the elementary symmetric functions embedding $\sigma$ (cf. Lemma 1.3) is schematically defined by the equations (1.3.2-1.3.3).*

*Proof.* We have a diagram locally

(1.7.7)
$$\begin{array}{ccc} H^m & \subset & \mathbb{P}^{m-1} \times \mathbb{A}^{2m} \times B \\ \downarrow & & \downarrow \\ X_B^{(m)} & \stackrel{\sigma}{\hookrightarrow} & \mathbb{A}^{2m} \times B. \end{array}$$

We have seen that the image of the top inclusion is defined by the equations (1.3.7), (1.4.14). The equations of the schematic image of $\sigma$ are obtained by eliminating the $Z$ coordinates from the latter equations, and this clearly yields the equations as claimed. □

Now as one byproduct of the proof of Theorem 1.1, we obtained generators of the ideal of the ordered half-discriminant $OD^m$. As a further consequence, we can determine the ideal of the discriminant locus $D^m$ in the symmetric product $X_B^{(m)}$ itself: let $\delta_m^x$ denote the discriminant of $F_0$, which, as is well known [5], is a polynomial in the $\sigma_i^x$ such that

(1.7.8)
$$\delta_m^x = G_1^2.$$

Set

(1.7.9)
$$\eta_{i,j} = \frac{(\sigma_m^y)^{i+j-2}}{t^{(i-1)(m-i)+(j-1)(m-j)}} \delta_x^m.$$

It is easy to see that this is a polynomial in the $\sigma_.^x$ and the $\sigma_.^y$, such that $\eta_{i,j} = G_i G_j$.

**Corollary 1.16.** *The ideal of $D^m$ is generated, locally near $mp$, by $\eta_{i,j}, i, j = 1, ..., m$.*

*Proof.* This follows from the fact that $\varpi_m$ is flat and that

$$\varpi_m^*(\eta_{i,j}) = G_i G_j, i, j = 1, ..., m$$

generate the ideal of $2OD^m = \varpi_m^*(D^m)$. □ □

*Proof of Lemma 1.2.* Consider the function $\eta_{i,i}$, a priori a rational function on $X_B^{(m)}$. Because it pulls back to the regular function $G_i^2$ on $X_B^{\lceil m\rceil}$, it follows that $\eta_{i,i}$ is in fact regular near the 'origin' $mp$. Clearly $\eta_{i,i}$ vanishes on the discriminant $D^m$. Now the divisor of $\eta_{i,i}$ on $U_i$ pulls back via $\varpi_m$ to the divisor of $G_i^2$, which also coincides with $\varpi_m^*(\Gamma^{(m)})$. Because $\varpi$ is finite flat, it follows the the divisor of $\eta_{i,i}$ coincides over $U_i$ with $\Gamma^{(m)}$ and in particular, the pullback of the ideal of $D^m$ has no embedded component in $U_i$. Since the $U_i, i = 1, ..., m$ cover a neighborhood of the exceptional locus in $X_B^{[m]}$, this shows $\mathfrak{c}_m^{-1}(D^m)$ has no embedded components, i.e. is Cartier, as claimed. The reader can check that the foregoing proof is logically independent of any results that depend on the statement of Lemma 1.2, so there is no vicious circle. □

Note that the ideal of the Cartier divisor $\mathfrak{c}_m^*(D^m)$ on $X_B^{[m]}$, that is, $\mathcal{O}_{X_B^{[m]}}(-\mathfrak{c}_m^*(D^m))$, is isomorphic in terms of our local model $\tilde{H}$ to $\mathcal{O}(2)$ (i.e. the pullback of $\mathcal{O}(2)$ from $\mathbb{P}^{m-1}$). This suggests that $\mathcal{O}(-\mathfrak{c}_m^*(D^m))$ is divisible by 2 as line bundle on $X_B^{[m]}$, as the following result indeed shows. First some notation. For a prime divisor $A$ on $X$, denote by $[m]_*(A)$ the prime divisor on $X_B^{[m]}$ consisting of schemes whose support meets $A$. This operation is easily seen to be additive, hence can be extended to arbitrary, not necessarily effective, divisors and thence to line bundles.

**Corollary 1.17.** *Set*

$$\mathcal{O}_{X_B^{[m]}}(1) = \omega_{X_B^{[m]}} \otimes [m]_*(\omega_X^{-1}). \tag{1.7.10}$$

*Then*

$$\mathcal{O}_{X_B^{[m]}}(-\mathfrak{c}_m^*(D^m)) \simeq \mathcal{O}_{X_B^{[m]}}(2) \tag{1.7.11}$$

*and*

$$\mathcal{O}_{X_B^{\lceil m\rceil}}(-o\mathfrak{c}_m^*(OD^m)) \simeq \varpi_m^*\mathcal{O}_{X_B^{[m]}}(1). \tag{1.7.12}$$

*Proof.* The Riemann-Hurwitz formula shows that the isomorphism (1.7.11) is valid on the open subset of $X_B^{[m]}$ consisting of schemes disjoint from the locus of fibre nodes of $\pi$. Since this open is big (has complement of codimension $> 1$), the iso holds on all of $X_B^{[m]}$. A similar argument establishes 1.7.12.. □

In practice, it is convenient to view (1.7.10) as a formula for $\omega_{X_B^{[m]}}$, with the understanding that $\mathcal{O}_{X_B^{[m]}}(1)$ coincides in our local model with the $\mathcal{O}(1)$ from the $\mathbb{P}^{m-1}$ factor, and that it pulls back over $X_B^{\lceil m\rceil} = X_B^{[m]} \times_{X_B^{(m)}} X_B^m$ to the $\mathcal{O}(1)$ associated to the blow up of the 'half discriminant' $OD^m$. We will also use the notation

$$\mathcal{O}(\Gamma^{(m)}) = \mathcal{O}_{X_B^{[m]}}(-1), \Gamma^{\lceil m\rceil} = \varpi_m^*(\Gamma^{(m)})$$

with the understanding that $\Gamma^{(m)}$ is Cartier, not necessarily effective, but $2\Gamma^{(m)}$ and $\Gamma^{\lceil m \rceil}$ are effective. Indeed $\Gamma^{(m)}$ is essentially never effective (compare Remark 2.15). Nonetheless, $-\Gamma^{(m)}$ is relatively ample on the Hilbert scheme $X_B^{[m]}$ over the symmetric product $X_B^{(m)}$, and will be referred to as the *discriminant polarization.*

1.8. **Globalization II.** We will now take up the globalization (over the base $B$, of arbitrary dimension) of the results of the previous section, in their '$Z$ coordinate' form, which is more closely related to the blowup structure compared to the $u, v$ coordinate form. We will do this for the ordered version of the Hilbert scheme, viz $X_B^{\lceil m \rceil}$ with its ordered cycle map to $X_B^m$. To this end, a key point is the globalization of the $G$ functions on $X_B^m$, or rather their divisors of zeros. We will see that these constitute a chain of $m$ essentially canonical 'intermediate diagonal' divisors, interpolating between the 'x-discriminant' and the 'y-discriminant'. These intermediate diagonals are Cartier divisors consisting of the big diagonal $OD^m$ plus certain boundary divisors, and the common schematic intersection of all of them is exactly $OD^m$. Most of our results on the intermediate diagonals are contained in the following statement.

**Proposition 1.18.** *Let $X/B$ be a flat family of nodal curves with irreducible base and generic fibre. Let $\theta$ be a relative node of $X/B$. Then*

(i) *there exists an analytic neighborhood $U$ of $\theta$ in $X$ and a rank-$m$ vector bundle $G^m(\theta)$, defined in $U_B^m \subset X_B^m$, together with a surjection in $U_B^m$:*

$$G^m(\theta) \to \mathcal{I}_{OD^m} \tag{1.8.1}$$

*giving rise to a natural polarized embedding*

$$U_B^{\lceil m \rceil} = \mathrm{B}\ell_{OD^m}(U_B^m) \hookrightarrow \mathbb{P}(\mathcal{I}_{OD^m}|_{U_B^m}) \to \mathbb{P}(G^m(\theta)). \tag{1.8.2}$$

(ii) *For any relatively affine, étale open $\tilde{U} \to U$ of $\theta$ in $X/B$ in which the 2 branches along $\theta$ are distinguishable, $G^m(\theta)$ splits over $(\tilde{U})^m$ as a direct sum of invertible ideals $G_j^m(\theta), j = 1, ..., m$;*

(iii) *Moreover if $V = U \setminus \pi^{-1}\pi(\theta)$, i.e. the union of the smooth fibres in $U$, the restriction of each $G_j^m(\theta)$ on $V_B^m$ is isomorphic to $\mathcal{I}_{OD^m}$.*

*Proof-construction.* Fix a boundary datum $(T, \delta, \theta)$ corresponding to $\theta$ as in §1.5. We first work locally in $X$, near a node in one singular fibre. Then we may assume the two branches along $\theta$ are distinguishable in $U$. We let $\beta_x, \beta_y$ denote the $x$ and $y$ branches, locally defined respectively by $y = 0, x = 0$. Consider the Weil divisor on $U_B^m$ defined by

$$OD_x^m(\theta) = OD^m + \sum_{i=2}^{m} \binom{i}{2} \sum_{\substack{I \subset [1, m] \\ |I| = i}} p_I^*(\beta_y) p_{I^c}^*(\beta_x). \tag{1.8.3}$$

**Claim 1.19.** *We have*

$$OD^m_x(\theta) = \mathrm{zeros}(G_1). \tag{1.8.4}$$

*where $G_1$ denotes the locally-defined Van der Monde determinant with respect to a local coordinate system as above.*

*Proof of claim.* Indeed each factor $x_a - x_b$ of $G_1$ vanishes on $p_I^*(\beta_y)p_{I^c}^*(\beta_x)$ precisely when $a, b \in I$; the rest is simple counting. $\square$

Thus, the divisor of $G_1$ is canonically defined, depending only on the choice of branch. Given this, it is natural in view of (1.7.1) to define the $j$-th *intermediate diagonal* along $\theta$ as

$$OD^m_{x,j}(\theta) = OD^m_x(\theta) + (j-1)\sum p_i^*(\beta_x) - (j-1)(m-j/2)\partial_\theta \tag{1.8.5}$$

where $\partial_\theta = \beta_x + \beta_y$ is the boundary divisor corresponding to the node $\theta$. Indeed (1.7.1) now shows

$$OD^m_{x,j}(\theta) = \mathrm{zeros}(G_j). \tag{1.8.6}$$

In particular, it is an effective Cartier divisor on $U^m_B$. Though each individual intermediate diagonal depends on the choice of branch, the collection of them does not. Indeed the elementary identity

$$\sigma^y_m V^m_1 = (-t)^{\binom{m}{2}} V^m_m. \tag{1.8.7}$$

shows that flipping $x$ and $y$ branches takes $-OD^m_{x,j}(\theta)$ to $-OD^m_{y,m+1-j}(\theta)$. Now set

$$G^m(\theta) = \bigoplus_{j=1}^{m} G^m_j(\theta), \quad \text{where} \quad G^m_j(\theta) = \mathcal{O}(-OD^m_{x,j}(\theta)) \tag{1.8.8}$$

This rank-$m$ vector bundle is independent of the choice of branch, as is the natural map $G^m(\theta) \to \mathcal{O}_{U^m_B}$. Therefore these data are defined globally over $B$ in a suitable analytic neighborhood of $\theta^m$. By Corollary 1.14, the image of this map is precisely the ideal of $OD^m$, i.e. there is a surjection

$$G^m(\theta) \to \mathcal{I}_{OD^m}|_{U^m_B} \to 0. \tag{1.8.9}$$

Applying the $\mathbb{P}$ functor, we obtain a closed embedding

$$\mathbb{P}(\mathcal{I}_{OD^m}) \to \mathbb{P}(G^m(\theta)) \tag{1.8.10}$$

Now the blow-up of the Weil divisor $OD^m$, which we have shown coincides with the Hilbert scheme $U^{\lceil m \rceil}_B$, is naturally a subscheme of $\mathbb{P}(\mathcal{I}_{OD^m})$, whence a natural embedding

$$U^{\lceil m \rceil}_B \to \mathbb{P}(G^m(\theta)) \tag{1.8.11}$$

Note that this is well-defined globally over $U^m_B$, which sits over a neighborhood of the boundary component in $B$ corresponding to $\theta$, i.e. $\partial_\theta$. $\square$

It is important to record here for future reference a compatibility between the $G^m_j(\theta)$ for different $m$'s. To this end let $U_x, U_y \subset U$ denote the complement of the $y$ (resp. $x$) branch, i.e. the open sets given by $x \neq 0, y \neq 0$.

**Lemma 1.20** (Localization formula). *We have for all* $1 \le j \le m - k_x - k_y$,

$$G^m_{j+k_y}(\theta)|_{U^{m-k_x-k_y}\times U_x^{k_x}\times U_y^{k_y}} = p^*_{U^{m-k_x-k_y}} G_j^{m-k_x-k_y}(\theta) \otimes \prod_{a<b>m-k_x-k_y} p^*_{a,b}(d_{a,b}) \tag{1.8.12}$$

*where* $d_{a,b}$ *is an equation for the Cartier divisor which equals the diagonal in* $a, b$ *coordinates; in divisor terms, this means*

$$OD^m_{x,j+k_y}|_{U^{m-k_x-k_y}\times U_x^{k_x}\times U_y^{k_y}} = p^*_{U^{m-k_x-k_y}}(OD_j^{m-k_x-k_y}) + \sum_{a<b>m-k_x-k_y} p^*_{a,b}(OD^2) \tag{1.8.13}$$

*where the last sum is Cartier and independent of* $j$.

*Proof.* We begin with the observation that, for the universal deformation $X_B$ of a node $p$, given by $xy = t$, the Cartesian square $X^2_B$ is nonsingular away from $(p, p)$, hence the diagonal is Cartier away from $(p, p)$, defined locally by $x_1 - x_2$ in the open set where $x_1 \neq 0$ or $x_2 \neq 0$ and likewise for $y$. Because the question is local and locally any deformation is induced by the universal one, a similar assertion holds for an arbitrary family. Now returning to our situation, let us write $n = m - k_x - k_y$ and $N, K_x, K_y$ for the respective index ranges $[1, n], [n+1, n+k_x], [n+k_x+k_y+1, m]$, and $x^N, y^N$ etc. for the corresponding monomials. Then $x_a - x_b$ is a single defining equation for the Cartier $(a, b)$ diagonal whenever $a$ or $b$ is in $K_x$. Then by (1.7.1), (1.7.2), we can write $G^m_{j+k_y}(\theta)$, up to a unit, i.e. a function vanishing nowhere in the open set in question, in the form

$$\frac{G_1^N (y^N)^{j-1}}{t^{(j-1)(n-j/2)}} \prod_{\substack{a<b \\ a \text{ or } b\in K_x}} (x_a - x_b) \frac{(y^N)^{k_y} \prod\limits_{a\in N, b\in K_y} (x_a - x_b)}{t^{nk_y}} \frac{\prod\limits_{a<b\in K_y} (x_a - x_b)}{t^{\binom{k_y}{2}}} \tag{1.8.14}$$

where we have used the fact that $\frac{(y^{K_x})^{j+k_y-1}}{t^{k_x(j+k_y-1)}}$ is a unit. Now in (1.8.14), the first factor is just $G^N_j$ while the second is the equation of a Cartier partial diagonal. The third factor is equal up to a unit to $\prod\limits_{a\in N, b\in K_y} (y_a - y_b)$, hence is also the equation of a Cartier partial diagonal. Finally, in the fourth factor, each subfactor $(x_a - x_b)/t = y_b^{-1} - y_a^{-1}$, so this too yields a Cartier partial diagonal. $\square$

Now recall the notion of boundary datum $(T, \delta, \theta)$ introduced in §1.5. We are now in position to determine globally the pullback of the intermediate diagonals to the partial normalization $X_T^\theta$:

**Corollary 1.21.** (i) *The pullback of $G^m_{j+k_y}(\theta)$ on $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$ extends over $U_B^{m-k_x-k_y} \otimes_B (X_T^\theta)_T^{\lceil k_x+k_y \rceil}$ to*

$$p^*_{U^{m-k_x-k_y}} G_j^{m-k_x-k_y}(\theta) \otimes p^*_{X_B^{\lceil k_x+k_y \rceil}} \mathcal{O}(-\Gamma^{\lceil k_x+k_y \rceil}) \otimes \bigotimes_{a \le m-k_x-k_y < b} p^*(\mathcal{O}(-OD^m_{a,b})) \tag{1.8.15}$$

*where the last factor is invertible;*

(ii) *the closure in $U_B^{m-k_x-k_y} \otimes_B (X_T^\theta)_T^{\lceil k_x+k_y \rceil}$ of the pullback of $OD^m_{j+k_y}(\theta)$ to $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$ equals*

$$p^*_{U^{m-k_x-k_y}} OD_j^{m-k_x-k_y}(\theta) + p^*_{X_B^{\lceil k_x+k_y \rceil}}(\Gamma^{\lceil k_x+k_y \rceil}) + \sum_{a \le m-k_x-k_y < b} p^*(OD^m_{a,b}) \tag{1.8.16}$$

*where each summand is Cartier.*

*Proof.* The first assertion is immediate from the Proposition. For the second, it suffices to note that the divisor in question has no components supported off $U_B^{m-k_x-k_y} \otimes_B U_B^{\lceil k_x+k_y \rceil}$. □

As an important consequence of this result, we can now determine the restriction of the $G$-bundles (i.e. the intermediate diagonals) on (essentially) the locus of cycles containing a node $\theta$ with given multiplicity; it is these restricted bundles that figure in the determination of the (polarized) node scrolls.

**Proposition 1.22.** *Let $(T, \delta, \theta)$ be a boundary datum, $1 \le j, n \le m$ be integers, and consider the map*

$$\mu^n : (X^\theta)_T^{\lceil k \rceil} \to X_B^m$$
$$\mu^n(z) = c_k(z) + n\theta.$$

*Then with $j_0 = \min(j, n)$, we have*

(1.8.17)
$$(\mu^n)^*(OD_j^m(\theta)) \sim -\binom{n-j_0+1}{2}\psi_x - \binom{j_0}{2}\psi_y + (n-j_0+1)\theta_x^{\lceil k \rceil} + (j_0-1)\theta_y^{\lceil k \rceil} + \Gamma^{\lceil k \rceil}$$

*where $\psi_x = \omega_{X_T^\theta/T} \otimes \mathcal{O}_{\theta_x}$ is the cotangent (psi) class at $\theta_x$ (which is a class from $T$, pulled back to $(X^\theta)_T^{\lceil k \rceil}$), $\theta_x^{\lceil k \rceil} = \sum\limits_{i=1}^{k} p_i^*(\theta_x)$ (which is a class from $(X^\theta)_T^k$, pulled back to $(X^\theta)_T^{\lceil k \rceil}$), and likewise for $y$.*

*Proof.* We factor $\mu^n$ through the map

$$\mu^n_{j_0} : (X^\theta)^{\lceil k\rceil}_T \to (X^\theta)^m_T$$
$$\mu^n_{j_0}(z) = ((\theta_x)^{n-j_0+1}, (\theta_y)^{j_0-1}, c_k(z)).$$

We may write $G^m_j$ as $G^n_{j_0}$ times a partial diagonal equation as above, and the inequalities on $j_0$ ensure that $G^n_{j_0}$ does not vanish identically on $\beta_x^{n-j_0+1} \times \beta_y^{j_0-1}$, where $\beta_x, \beta_y$ are the branch neighborhoods of $\theta_x, \theta_y$ in $X^\theta_T$. Then it is straightforward that the last two summands in (1.8.16) correspond to the last three summands in (1.8.17), e.g. a diagonal $D^m_{a,b}$ with $a \leq n - j_0 + 1$ coincides with $p_b^*\theta_x$. So it's just a matter of evaluating the pullback of $OD^n_j(\theta)$. For the latter, we use a Laplace (block) expansion of $G_{j_0}$ on the first $n - j_0 + 1$ rows. In this expansion, the leading term is the first, i.e. the product of the two corner blocks. There writing $x_a - x_b = dx$, a generator of $\psi_x$, and likewise for $\psi_y$, we get the asserted form as in (1.8.17). □

## 2. The tautological module

In this section we will compute arbitrary powers of the discriminant polarization $\Gamma^{(m)}$ on the Hilbert scheme $X^{[m]}_B$. The computation will be a by-product of a stronger result determining the (additive) *tautological module* on $X^{[m]}_B$, to be described informally in this introduction, and defined formally in the body of the chapter (see Definition 2.31).

The tautological module

$$T^m = T^m(X/B) \subset A^\cdot(X^{[m]}_B)_\mathbb{Q}$$

is to be defined as the $\mathbb{Q}$-vector space generated by certain basic *tautological classes* (as described below). On the other hand, let

$$\mathbb{Q}[\Gamma^{(m)}] \subset A^\cdot(X^{[m]}_B)_\mathbb{Q}$$

be the subring of the Chow ring generated by the discriminant polarization. Then the main result of this chapter is

**Theorem 2.1** (Module Theorem). *Under intersection product, $T^m$ is a $\mathbb{Q}[\Gamma^{(m)}]$-module; moreover, multiplication by $\Gamma^{(m)}$ can be desribed explicitly.*

Because $1 \in T^m$ by definition, this statement includes the nonobvious assertion that

$$\mathbb{Q}[\Gamma^{(m)}] \subset T^m;$$

in other words, any polynomial in $\Gamma^{(m)}$ is (explicitly) tautological. In this sense, the Theorem includes an 'explicit' (in the recursive sense, at least) computation of all the powers of $\Gamma^{(m)}$.

Now the aforementioned basic tautological classes come in two main flavors (plus some subflavors).

(i) The (classes of) *(relative) diagonal loci* $\Gamma^{(m)}_{(n_1,n_2,...)}$: this locus is essentially the closure of the set of schemes of the form $n_1p_1 + n_2p_2 + ...$ where $p_1, p_2...$ are distinct smooth points of the same (arbitrary) fibre.

More generally, we will consider certain 'twists' of these, denoted $\Gamma_{(n_1,n_2,...)}[\alpha_1, \alpha_2...]$, where the $\alpha.$ are 'base classes', i.e. cohomology classes on $X$.

(ii) The *node classes*. First, the *node scrolls* $F^n_j(\theta)$: these are, essentially, $\mathbb{P}^1$-bundles over an analogous diagonal locus $\Gamma^{(m-n)}_{(n.)}$ associated to a boundary family $X^\theta_T$ of $X_B$, whose general fibre can be naturally identified with the punctual Hilbert scheme component $C^n_j$ along the node $\theta$.

Additionally, there are the *node sections*: these are simply the classes $-\Gamma^{(m)}.F$ where $F$ is a node scroll as above (the terminology comes from the fact that $\Gamma^{(m)}$ restricts to $\mathcal{O}(1)$ on each fibre of a node scroll).

All these classes again admit twisted versions, essentially obtained by multiplying by bases classes from $X$.

Effectively, the task of proving Theorem 2.1 has two parts.

(i) Express a product $\Gamma^{(m)}.\Gamma_{(n.)}$ in terms of other diagonal loci and node scrolls, see Proporsition 2.16.

(ii) For each node $\theta$ and associated ($\theta$-normalized) boundary family $X^\theta_T$, determine a series of explicit line bundles $E^n_j(\theta), j = 1, ..., n$ on the relative Hilbert scheme $(X^\theta_T)^{[m-n]}_T$ together with an identification

$$F^n_j(\theta) \simeq \mathbb{P}(E^n_j(\theta) \oplus E^n_{j+1}(\theta)),$$

such that the restriction of the discriminant polarization $-\Gamma^{(m)}$ on $F^n_j(\theta)$ becomes the standard $\mathcal{O}(1)$ polarization on the projectivized vector bundle (see Proposition 2.24); in fact, $E^n_j(\theta)$ is just the sum of the polarization $\Gamma^{[m-n]}$ and a suitable base divisor, see (2.4.21), (2.4.22). It then transpires that the restriction of an arbitrary power $(\Gamma^{(m)})^k$ on $F$ can be easily and explicitly expressed in terms of other node classes coming from the Chern classes of $E$ (see Corollary 2.25).

2.1. **The small diagonal.** We begin our study of diagonal-type loci and their intersection product with the discriminant polarization with the smallest such locus, i.e. the small diagonal. In a sense this is actually the heart of the matter, which is hardly surprising, considering as the small diagonal is in the 'most special' position vis-a-vis the discriminant. The next result is in essence a corollary to the Blowup Theorem 1.1.

Let $\Gamma_{(m)} \subset X^{[m]}_B$ be the small diagonal, which parametrizes schemes with 1-point support, and which is the pullback of the small diagonal

$$D_{(m)} \simeq X \subset X^{(m)}_B.$$

The restriction of the cycle map yields a birational morphism

$$\mathfrak{c}_m : \Gamma_{(m)} \to X$$

which is an isomorphism except over the nodes of $X/B$. Fix a covering system of boundary data $\{(T., \delta., \theta.)\}$ and focus on its typical node $\theta$. Let

$$J_m^{\theta.} = \bigcap_i J_m^{\theta_i} \subset \mathcal{O}_X$$

be the ideal sheaf whose stalk at each fibre node $\theta_i$ is locally of type $J_m$ as in §0. Note that $J_m^{\theta.}$ is well-defined independent of the choice of local parameters and independent as well of the ordering of the branches at each node, hence makes sense and is globally defined on $X$.

**Proposition 2.2.** *Via $\mathfrak{c}_m$, $\Gamma_{(m)}$ is equivalent to the blow-up of $J_m^{\theta.}$. If $\mathcal{O}_{\Gamma_{(m)}}(1)_J$ denotes the canonical blowup polarization, we have*

$$\mathcal{O}_{\Gamma_{(m)}}(-\Gamma^{(m)}) = \omega_{X/B}^{\otimes\binom{m}{2}} \otimes \mathcal{O}_{\Gamma_{(m)}}(1)_J. \tag{2.1.1}$$

*Furthermore, if $X$ is smooth at a node $\theta$, then $\Gamma_{(m)}$ has multiplicity $\min(i, m-i)$ along the corresponding divisor $C_i^m - \{Q_i^m, Q_{i+1}^m\}$ for $i = 1, ..., m-1$. In particular, $\Gamma_{(m)}$ is smooth along $(C_1^m - Q_2^m) \cup (C_{m-1}^m - Q_{m-1}^m)$.*

*Proof.* We may work with the ordered versions of these objects, then pass to $\mathfrak{S}_m$-invariants. We fist work locally over a neighborhood of a point $p^m \in X_B^m$ where $p$ is a fibre node. We may then assume $X$ is a smooth surface and $X/B$ is given by $xy = t$, as the general case is derived from this by base-change. Then the ideal of $OD^m$ is generated by $G_1, ..., G_m$ and $G_1$ has the Van der Monde form $v_x^m$, while the other $G_i$ are given by (1.7.1). We try to restrict the ideal of $OD^m$ on the small diagonal $OD_{(m)}$. To this end, note to begin with the natural map

$$\mathcal{I}_{OD^m} \to \omega^{\binom{m}{2}}, \omega := \omega_{X/B}.$$

Indeed this map is clearly defined off the singular locus of $X_B^m$, hence by reflexivity of $\mathcal{I}_{OD^m}$ extends everywhere, hence moreover factors through a map

$$\mathcal{I}_{OD^m}.OD_{(m)} = \mathcal{I}_{OD^m} \otimes \mathcal{O}_{OD_{(m)}}/(\text{torsion}) \to \omega^{\binom{m}{2}}.$$

To identify the image, note that

$$(x_i - x_j)|_{OD_{(m)}} = dx = x\frac{dx}{x}$$

and $\eta = \frac{dx}{x} = -\frac{dy}{y}$ is a local generator of $\omega$ along $\theta$. Therefore

$$G_1|_{OD_{(m)}} = x^{\binom{m}{2}}\eta^{\binom{m}{2}}.$$

From (1.7.2) we then deduce

$$G_i|_{\Gamma_{(m)}} = x^{\binom{m-i+1}{2}}y^{\binom{i}{2}}\eta^{\binom{m}{2}}, i = 1, ..., m. \tag{2.1.2}$$

Since $G_1, ..., G_m$ generate the ideal $I_{OD^m}$ along $\theta$, it follows that over a neighborhood of $\theta$, we have

$$I_{OD^m}.OD_{(m)} \simeq J_m^\theta \otimes \omega^{\binom{m}{2}}.$$

This being true for each node, it is also true globally. Consequently, passing to the $\mathfrak{S}_m$-quotient, we also have

$$I_{D^m}.D_{(m)} \simeq J_m^\theta \otimes \omega^{\binom{m}{2}}.$$

Then pulling back to $X_B^{[m]}$ we get (2.1.1).

Finally, it follows from the above, plus the explicit description of the model $H_m$, that, along the 'finite' part $C_i^m - Q_{i+1}^m$, $\Gamma_{(m)}$ has equation $x^{m-i} - uy^i$ where $u$ is an affine coordinate on $C_i^m - Q_{i+1}^m$, from which our last assertion follows easily. $\square$

Now it follows from the Proposition that, given a node $\theta$ of $X/B$, the pullback ideal of $J_m^\theta$ on $\Gamma = D_{(m)}$ is an invertible ideal supported on the inverse image of $\theta$, i.e. $\bigcup_{i=1}^{m-1} C_i^m(\theta)$; we denote this ideal by $\mathcal{O}_\Gamma(1)_{J_m^\theta}$ or $\mathcal{O}_\Gamma(-e_m^\theta)$. It must not be confused with the pullback of the reduced ideal of $\theta$.

**Proposition 2.3.** *We have*

$$e_m^\theta = \sum_{i=1}^{m-1} \beta_{m,i} C_i^m(\theta) \tag{2.1.3}$$

*where the $\beta_{m,i}$ are as in §0.*

*Proof.* We may fix $\theta$ and work locally with the universal family $xy - t$. Clearly the support of $e_m$ is $C^m = \bigcup_{i=1}^{m-1} C_i^m$, so we can write

$$e_m = \sum_{i-1}^{m-1} b_{m,i} C_i^m$$

and we have

$$-e_m^2 = \deg(\mathcal{O}(1).e_m) = \sum_{i=1}^{m-1} b_{m,i} =: b_m.$$

Now the general point on $C_i^m$ corresponds to an ideal $(x^{m-i} + ay^i), a \in \mathbb{C}^*$ and the rational function $x^{m-i}/y^i$ restricts to a coordinate on $C_i^m$. It follows that if $A_i \subset X$ is the curve with equation $f_i = x^{m-i} - ay^i$ for some constant $a \in \mathbb{C}^*$, then its proper transform $\tilde{A}_i$ meets $C^m$ transversely in the unique point $q \in C_i^m$ with coordinate $a$, so that

$$\tilde{A}_i.e_m = b_{m,i}.$$

Thus, setting $J_{m,i} = J_m + (f_i)$ we get following characterization of $b_{m,i}$:

$$b_{m,i} = \ell(\mathcal{O}_X/J_{m,i}).$$

To compute this, we start by noting that a cobasis $B_m$ for $J_m$, i.e. a basis for $\mathcal{O}_X/J_m$ is given by the monomials $x^ay^b$ where $(a,b)$ is an interior point of the polygon $S_m$ as in §0; equivalently, the square with bottom left corner $(a,b)$ lies in $R_m$. Then a cobasis $B_{m,i}$ for $J_{m,i}$ can be obtained by starting with $B_m$ and eliminating

- all monomials $x^ay^b$ with $b\geq i$;
- for any $j$ with $\binom{j}{2}\geq i$, all monomials that are multiples of $x^{\binom{m+1-j}{2}+m-i}y^{\binom{j}{2}-i}$;

the latter of course comes from the relation
$$x^{\binom{m+1-j}{2}}y^{\binom{j}{2}}\equiv 0 \mod J_m.$$
Graphically, this cobasis corresponds exactly to the polygon $S_{m,i}$ in §0, hence
$$b_{m,i}=\beta_{m,i}, b_m=\beta_m;$$
□

**Corollary 2.4.** *Suppose $B$ is 1-dimensional. With the above notations, we have*
$$e_m^2=-\sigma\beta_m, \tag{2.1.4}$$
*where $\sigma$ is the number of nodes of $X/B$;*
$$\Gamma^{(m)}.\Gamma_{(m)}=\sum\beta_{m,i}C_i^m-\binom{m}{2}\omega_{X/B}; \tag{2.1.5}$$
$$\int\limits_{\Gamma_{(m)}}(\Gamma^{(m)})^2=-\sigma\beta_m+\binom{m}{2}^2\omega_{X/B}^2. \tag{2.1.6}$$
.

*Remark* 2.5. The components $C_i^m(\theta), i=1,...,m-1$ of $e_m$ are $\mathbb{P}^1$-bundles over $\theta$ and are special cases of the node scrolls, encountered in the previous section, and which will be further discussed in §2.3 below. The coefficients $\beta_{m,i}$ play an essential role our intersection calculus.

For the remainder of the paper, we set
$$\omega=\omega_{X/B}.$$
We will view this interchangeably as line bundle or divisor class.

2.2. **Monoblock and polyblock digaonals: ordered case.** Returning to our family $X/B$ of nodal curves, we now begin extending the results of §2.1 to the more general diagonal loci as defined above, first for those that live over all of $B$, and subsequently for loci associated to the boundary. We call these *monoblock* and *polyblock diagonals*, depending on whether they correspond to a single block or to a partition. These loci come in both ordered and unordered versions, the ordered version being more convenient, the unordered one the more 'correct' or

natural one. We begin with the ordered monoblock diagonal, defined as follows. For an index-set $I \subset [1, m]$, set

$$OD_I^m = OD_I = p_I^{-1}(OD_{|I|}) \subset X_B^m, \tag{2.2.1}$$

where $p_I : X_B^m \to X_B^{|I|}$ is the projection and $OD_{|I|} \subset X_B^{|I|}$ is the small diagonal; thus, $OD_I^m$ is $X_B^{(I.)}$ where $(I.)$ is any partition on $[1, m]$ equivalent to the single block $I$. Note that

$$OD^m = \sum_{1 \leq a < b \leq m} OD_{a,b}^m. \tag{2.2.2}$$

Also, we have an isomorphism $OD_I^m \simeq X_B^{m-|I|+1}$. Let

$$\Gamma_I = \Gamma_I^{\lceil m \rceil} := oc^{-1}(OD_I) \subset X_B^{\lceil m \rceil} \tag{2.2.3}$$

These are called (ordered) monoblock diagonal loci (by comparision, the unordered monoblock diagonals, to be studied below, will be associated to a block *size* rather than a block). . Note that $OD_I$, hence $\Gamma_I$, are defined locally near a node by equations

$$x_i - x_j = 0 = y_i - y_j, \ \forall i, j \in I. \tag{2.2.4}$$

Similarly, for any partition

$$(I.) = (I_1, ..., I_r) \subset [1, m],$$

we define an analogous locus ( ordered *polyblock diagonal* )

$$\Gamma_{I_1|...|I_r} = \Gamma_{I_1|...|I_r}^{\lceil m \rceil} \subset X_B^{\lceil m \rceil} \tag{2.2.5}$$

and note that

$$\Gamma_{I_1|...|I_r} = \Gamma_{I_1} \cap ... \cap \Gamma_{I_r} \tag{2.2.6}$$

(transverse intersection). Also

$$\Gamma_{(I.)} = oc^{-1}(OD_{(I.)}) \tag{2.2.7}$$

where $OD_{(I.)} \subset X_B^m$ is the analogous polyblock diagonal. Note that when $(I.)$ is full of length $r$, we have

$$OD_{(I.)} \simeq X_B^r \tag{2.2.8}$$

Now to analyze a monoblock diagonal locus $OD_I$, a key technical question is to determine the part of $OD_I$ over the boundary of $B$; or equivalently, fixing a boundary datum $(\theta, T, \delta)$, with the associated map $\phi : X_T^\theta \to X$, to determine $(\phi^m)^*(OD_I)$. Fixing such a boundary datum, the answer is as follows, where we denote the $x$ and $y$ axes in $X_T^\theta$ by $X', X''$ respectively.

**Lemma 2.6.** *The irreducible components of the boundary of $\Gamma_I$ over $T$ are as follows:*

(i) *for each index-set* $K$*,* $[1,m] \supset K \supset I$*, a locus* $\tilde{\Theta}_{K/I}$*, mapping birationally to its image* $\Theta_{K/I} \subset OD_I$*;*

(ii) *for each* $K \subset I^c = [1,m] \setminus I$*, ditto;*

(iii) *for each* $K$ *straddling* $I$ *and* $I^c$*, and each* $j = 1, ..., |I| - 1$*, a component* $OF_j^{I:K-I|K^c-I}(\theta) \subset OF_j^I(\theta)$ *projecting as* $\mathbb{P}^1$*-bundle to its image in* $(X_T^\theta)^{\lceil m-|I| \rceil}$*, which lies over* $(X')^{K-I} \times_T (X")^{K^c-I} =: (X_T^\theta)^{K-I|K^c-I} \subset (X_T^\theta)^{m-|I|}$*.*

*Proof.* We may fix a node $\theta$ and work locally over a neighborhood of $\theta$ in $X$. From the definition and basic properties of node scrolls (see §1.5), the main point is to determine the boundary of $OD_I$. But this is easily determined: referring to (1.7.5), the latter boundary is given locally by

$$\bigcup_{K \subset [1,m]} OD_I \cap \Theta_K.$$

Set $\Theta_{K/I} = OD_I \cap \Theta_K$. To describe these, there are 3 cases depending on $K$:

(i) if $I \subset K$, then
$$\Theta_{K/I} = (X')^{K/I} \times (X")^{K^c};$$

(ii) if $I \subset K^c$, then
$$\Theta_{K/I} = (X')^K \times (X")^{K^c/I};$$

(iii) otherwise, i.e. if $I$ straddles $K$ and $K^c$, then
$$\begin{aligned}\Theta_{K/I} &= \{y_i = 0, \forall i \in K \cup I, x_i = 0, \forall i \in K^c \cup I\} \\ &= (X')^{K-I} \times (X")^{K^c-I} \times 0^I =: X^{K-I|K^c-I}\end{aligned}$$

(to specify the special value $s \in B$, a subscript $s$ may subsequently be added in the above).

Now is an elementary check that the loci of type (i) and (ii) are precisely the irreducible components of the special fibre of $OD_I$, while the union of the loci $\Theta_{K/I}$ of type (iii) coincides with the intersection of $OD_I$ with the fundamental locus (=image of exceptional locus) of the ordered cycle map $oc_m$, i.e. the locus of cycles containing the node with multiplicity $> 1$. Also, each $\Theta_{K/I}$ of type (iii) is of codimension 2 in $OD_I$. On the other hand, each such $\Theta_{K/I} = X^{K-I|K^c-I}$ is just a component of the invese image in $X_B^m$ of the locus denoted $X^{(a,b)}$ in Lemma 1.4.15, where $a = |K - I|, b = |K^c - I|$, and therefore by that Lemma, the ordered cycle map over it is a union of $\mathbb{P}^1$ bundles, viz

$$oc_m^{-1}(X^{K-I|K^c-I}) = \bigcup_{j=1}^{|I|-1} OF_j^{I:K-I|K^c-I} \tag{2.2.9}$$

where $OF_j^{I:K-I|K^c-I}$ is the pullback of $F_j^{(m-a-b:a|b)}$ over $X^{K-I|K^c-I}$, which is a $\mathbb{P}^1$ bundle with fibre $C_j^{|I|}$. This concludes the proof. □

Notice that, given disjoint index-sets $K_1, K_2$ with $K_1 \coprod K_2 = I^c$, the number of straddler sets $K$ such that $K - I = K_1, K^c - I = K_2$ is precisely $2^n - 2$ (i.e. the number of proper nonempty subsets of $I$). Thus, a given $OF_j^{I:K_1|K_2}$ will lie on this many components of $\tilde{\Theta}$. This however is a completely separate issue from the multiplicity of $OF_j^{I:K_1|K_2}$ in the intersection cycle $\Gamma^{[m]}.\Gamma_I$, which has to do with the blowup structure and will be determined below.

From the foregoing analysis, we can easily compute the intersection of a monoblock diagonal cycle with the discriminant polarization, as follows. We will fix a covering system of boundary data $(T_s, \delta_s, \theta_s)$,and recall that each datum must be weighted by $\frac{1}{\deg(\delta_s)}$.

**Proposition 2.7.** *We have an equality of divisor classes on $\Gamma_I$:*

$$\Gamma^{\lceil m \rceil}.\Gamma_I = \sum_{i<j\notin I} \Gamma_{I|\{i,j\}} + |I| \sum_{i\notin I} \Gamma_{I\cup\{i\}} \tag{2.2.10}$$
$$-\binom{|I|}{2} p^*_{\min(I)}\omega + \sum_s \frac{1}{\deg(\delta_s)} \sum_{j=1}^{|I|-1} \beta_{|I|,j}\delta^I_{s,j*}OF^I_j(\theta_s),$$

*where $I|\{i,j\}$ and $I \cup \{i\}$ denote the evident diblock partition and uniblock, respectively, the 4th term denotes the class of the image of the node scroll on $\Gamma_I$, $OF^I_j(\theta) = \sum\limits_{K_1\coprod K_2=I^c} OF_j^{I:K_1|K_2}(\theta)$, and $\delta^I_{s,j}$ is the natural map of the latter to $\Gamma_I \subset X_B^{[m]}$ ; precisely put, the line bundle on $\Gamma_I$ given by $\mathcal{O}_{\Gamma_I}(\Gamma^{\lceil m \rceil}) \otimes p^*_{\min(I)}(\omega^{|I|})$ is represented by an effective divisor comprising the 1st, 2nd and 4th terms of the RHS of (2.2.10).*

*Proof.* To begin with, the asserted equality trivially holds away from the exceptional locus of $oc_m$, where the 1st, second and third summands come from components $\Gamma_{i,j}$ of $\Gamma^{\lceil m \rceil}$ having $|I \cap \{i,j\}| = 0, 1, 2$, respectively.

Next, both sides being divisors on $\Gamma_I$, it will suffice to check equality away from codimension 2, e.g. over a generic point of each (boundary) locus $(X_T^\theta)^{K-I|K^c-I}$. But there, our cycle map $oc_m$ is locally just $oc_r \times \mathrm{iso}$, $r = |I|$, with

$$\Gamma^{\lceil m \rceil} \sim \Gamma^{\lceil r \rceil} + \sum_{\{i,j\}\not\subset I} \Gamma_{i,j}.$$

We are then reduced to the case of the small diagonal, discussed in the previous subsection.

□

The extension of this result from the monoblock to the polyblock case- still in the ordered setting- is in principle straightforward, but a bit complicated to describe. Again, a key issue is to describe the boundary of a polyblock diagonal locus $OD_{(I.)}$ in terms of the decomposition (1.7.5). Fix a boundary datum $(T, \delta, \theta)$. To simplify notations, we will assume, losing no generality, that the partition $I.$ is

full. Now consider an index-set $K \subset [1,m]$. As before, $K$ is said to be a *straddler* with respect to a block $I_\ell$ of $(I.)$, and $I_\ell$ is a *straddler block* for $K$, if $I_\ell$ meets both $K$ and $K^c$. The *straddler number* $\mathrm{strad}_{(I.)}(K)$ of $K$ w.r.t. $(I.)$ is the number of straddler blocks $I_\ell$. The *straddler portion* of $(I.)$ relative to $K$ is by definition the union of all straddler blocks, i.e.

(2.2.11)
$$s_K(I.) = \bigcup_{I_\ell \cap K \neq \emptyset \neq I_\ell \cap K^c} I_\ell.$$

The $x$- (resp. $y$-)*-portion* of $(I.)$ (relative to $K$, of course) are by definition the partitions

(2.2.12)
$$x_K(I.) = \{I_\ell : I_\ell \subset K\}, y_K(I.) = \{I_\ell : I_\ell \subset K^c\}.$$

Finally the *multipartition data* associated to $(I.)$ w.r.t. $K$ are

(2.2.13)
$$\Phi_K(I.) = (s_K(I.) : x_K(I.)|y_K(I.)).$$

In reality, this is a partition broken up into 3 parts: the *nodebound* part $s_K(I.)$, a single block, plus 2 *at large* parts, an $x$ part and a $y$ part. As before, we set

(2.2.14)
$$X^{\Phi_K(I.)} = (X')^{x_K(I.)} \times (X")^{y_K(I.)}$$

and equip it as before with the map to $X^m_s$ obtained by inserting the node $\theta$ at the $s_K(I.)$ positions.Now the analogue of Lemma 2.6 is the following

**Lemma 2.8.** *For any partition $(I.)$ and boundary datum $(T, \delta, \theta)$,, the corresponding boundary portion of $\Gamma_{(I.)}$ is*

(2.2.15)
$$\bigcup_{\mathrm{strad}_{(I.)}(K)=0} \tilde{\Theta}_{K,(I.)} \cup \bigcup_{\ell} \bigcup_{I'.\coprod I".=I.\setminus I_\ell} \bigcup_{j=1}^{|I_\ell|-1} OF_j^{(I_\ell:I'.|I".)}(\theta)$$

*Proof.* Now, one can easily verify

(2.2.16)
$$OD_{(I.)} \cap \Phi_K = X^{\Phi_K(I.)} =: \Theta_{K,(I.)}$$

so that

(2.2.17)
$$OD_{(I.)} \cap X^m_0 = \bigcup_{K \subset [1,m]} \Theta_{K,(I.)}.$$

Now, an elementary observation is in order. Clearly, the codimension of $OD_{(I.)}$ in $X^m_B$ is $\sum\limits_\ell (|I_\ell| - 1)$, and this also equals the codimension of $OD_{(I.)} \cap X^m_0$ in $X^m_0$. On the other hand, we have

(2.2.18)
$$\dim(\Theta_{K,(I.)}) = m - \left( \sum_{I_\ell \text{ nonstraddler rel} K} (|I_\ell| - 1) + \sum_{I_\ell \text{ straddler rel} K} |I_\ell| \right)$$
$$= m - \sum_\ell (|I_\ell| - 1) - \mathrm{strad}_{(I.)}(K).$$

It follows that

- the index-sets $K$ such that $\Theta_{K,(I.)}$ is a component of the boundary $OD_{(I.)} \cap (X^\theta_T)^m$ are precisely the nonstraddlers;
- those $K$ such that $\Theta_{K,(I.)}$ is of codimension 1 in the special fibre are precisely those of straddle number 1 (unistraddlers).

Next, what are the preimages of these loci upstairs in the ordered Hilbert scheme $X_B^{[m]}$? They can be analyzed as in the monoblock case:

- if $K$ is a nonstraddler, a general cycle parametrized by $\Theta_{K,(I.)}$ is disjoint from the node, so there will be a unique component $\tilde{\Theta}_{K,(I.)} \subset oc_m^{-1}(\Theta_{K,(I.)}$ dominating $\Theta_{K,(I.)}$;
- if $K$ is a unistraddler (straddle number $= 1$), the dominant components of $oc_m^{-1}(\Theta_{K,(I.)})$ will be the $\mathbb{P}^1$-bundles $F_j^{\Phi_K(I.)}, j = 1, , , s_K(I.) - 1$; note that if $I_\ell$ the unique block making $K$ a straddler, then $\Phi_K(I.) = (I_\ell : x_K(I.)|y_K(I.))$; moreover as $K$ runs through all unistraddlers, $\Phi_K(I.)$ runs through the date consisting of a choice of block $I_\ell$ plus a partition of the set of remaining blocks in two ('$x$- and $y$-blocks');
- because all fibres of $oc_m$ are at most 1-dimensional, while every component of the boundary is of codimension 1 in $\Gamma_{(I.)}$, no index-set $K$ with straddle number $\mathrm{strad}_{(I.)}(K) > 1$ (i.e. multistraddler) can contribute a component to that special fibre.

This completes the proof. $\square$

What the Lemma means is that the analysis leading to Proposition 2.7 extends with no essential changes to the polyblock case, and therefore the natural analogue of that Proposition holds. This is the subject of the next Corollary which for convenience will be stated in slightly greater generality to allow for twisting. Let us identify

(2.2.19) $$\Gamma_{(I.)} \simeq \prod_{j=1}^{r} {}_BX$$

where $I_1, ..., I_r$ are the blocks. For a collection $\alpha_1, ..., \alpha_r$ of cohomology classes on $X$, recall from Section 0.2.1 the notation

(2.2.20) $$\Gamma_{(I.)} \star_t [\alpha_1, ..., \alpha_r] = s_t(p_1^*(\alpha_1), ..., p_r^*(\alpha_r)$$

where $s_t$ are the elementary symmetric functions. Similarly, for any subvariety (or homology class) $Y$ on $\Gamma_{(I.)}$, we have

(2.2.21) $$Y \star_t [\alpha_1, ..., \alpha_r] = \Gamma_{(I.)} \star_t [\alpha.] \cup [Y].$$

When $t = r$, we write $Y \star_r [\alpha.]$ simply as $Y[\alpha.]$. We will use this in particular when $Y = OF_j^{I_\ell}(\theta)$ and note that, because $OF_j^{I_\ell}$ projects to a section (viz. $\theta$ in each of

the $I_\ell$ coordinates, we have

$$\deg(\alpha_\ell) \geq \dim(B) \Rightarrow OF_j^{I_\ell}[\alpha.] = 0. \tag{2.2.22}$$

To state our result compactly, it will be convenient to introduce the following operations on partitions :

$$U_{k,\ell}(I.) = (..., I_k \cup I_\ell, ..., \widehat{I_\ell}, ...) \tag{2.2.23}$$

(i.e. uniting the $k$th and $\ell$th blocks),

$$V_\ell(I., i) = (..., I_\ell \cup i, ...). \tag{2.2.24}$$

**Corollary 2.9.** (i) *For any partition $I. = I_1|...|I_r$ on $[1, m]$, we have an equality of divisor classes on $\Gamma_{I.}$:*

$$\begin{aligned}\Gamma^{\lceil m \rceil}.\Gamma_{(I.)} = &\sum_{i<j\notin\bigcup I.} \Gamma_{(I.|\{i,j\})} + \sum_\ell |I_\ell| \sum_{i\notin\bigcup I.} \Gamma_{(V_\ell(I.,i))} \\ &+ \sum_{j<\ell} |I_j||I_\ell|\Gamma_{(U_{k,\ell}(I.))} - \Gamma_{(I.)} \star_1 [\binom{|I_1|}{2}\omega, ...\binom{|I_s|}{2}\omega] + \\ &\sum_s \frac{1}{\deg(\delta_s)} \sum_\ell \sum_{j=1}^{|I_\ell|-1} \beta_{|I_\ell|,j}\delta^{I_\ell}_{s,j*}[OF_j^{I_\ell/I.}(\theta_s)]\end{aligned} \tag{2.2.25}$$

*where*

$$[OF_j^{I_\ell/I.}(\theta)] = \sum_{I'.\coprod I".=I.\setminus I_\ell} \sum_{j=1}^{|I_\ell|-1} \delta^{I_\ell}_{s,j*}[OF_j^{(I_\ell:I'.|I".)}(\theta)]. \tag{2.2.26}$$

(ii) *if $I.$ is full, we have*

$$\begin{aligned}\Gamma^{\lceil m \rceil}.\Gamma_{(I.)} = &\sum_{j<\ell} |I_j||I_\ell|\Gamma_{(U_{j,\ell}(I.))} \\ &- \Gamma_{(I.)} \star_1 [\binom{|I_1|}{2}\omega, ...\binom{|I_s|}{2}\omega] + \sum_s \frac{1}{\deg(\delta_s)} \sum_\ell \sum_{j=1}^{|I_\ell|-1} \beta_{|I_\ell|,j}\delta^{I_\ell}_{s,j*}[OF_j^{I_\ell/I.}(\theta_s)]\end{aligned} \tag{2.2.27}$$

(iii) *if $I.$ is full, we have more generally*

$$\begin{aligned}\Gamma^{\lceil m \rceil}.\Gamma_{(I.)}[\alpha.] = &\sum_{j<\ell} |I_j||I_\ell|\Gamma_{(U_{j,\ell}(I.))}[..., \alpha_j ._X \alpha_\ell, ...\widehat{\alpha_\ell}...] \\ &-\Gamma_{(I.)}[\alpha.] \star_1 [\binom{|I_1|}{2}\omega, ...\binom{|I_s|}{2}\omega] \\ &+ \sum_s \frac{1}{\deg(\delta_s)} \sum_\ell \sum_{j=1}^{|I_\ell|-1} \beta_{|I_\ell|,j}\delta^{I_\ell}_{s,j*}OF_j^{I_\ell/I.}(\theta_s)[\alpha.]\end{aligned} \tag{2.2.28}$$

□

2.3. **Monoblock and polyblock diagonals: unordered case.** We need the analogues of the formulae of the latter section in the (unordered) Hilbert scheme. These are essentially straightforward, and may be obtained from the ordered versions using push-forward by the symmetrization map $\varpi_m$. We begin with the monoblock case. Recall first the the monoblock (unordered) diagonal $\Gamma_{(n)}$ is defined as a set by

$$\Gamma_{(n)} = \varpi_m(\Gamma_{(I)})$$

for any block $I$ of cardinality $n$. More generally, we may similarly define $\Gamma_{(n.)}$ for any distribution $(n.)$: this will be considered in detail below. Similarly, if $\alpha$ is any cohomology class on $X$, there is an associated class $X_B^{(n|1.)}[\alpha]$ on the symmetric product $X_B^{(m)}$, and we define

$$\Gamma_{(n)}[\alpha] = c^*(X_B^{(n|1.)}[\alpha]).$$

alternatively, this could also be defined as

$$\Gamma^{(n)}[\alpha] = \frac{1}{(m-n)!}\varpi_{m*}(\Gamma_{(I)})[\alpha].$$

Note the following elementary facts:

(i)

(2.3.1) $$\varpi_{m*}(\Gamma^{\lceil m\rceil}.\Gamma_I) = \Gamma^{(m)}.\varpi_{m*}\Gamma_I$$

(projection formula, because $\varpi_m^*(\Gamma^{(m)}) = \Gamma^{\lceil m\rceil}$; NB $\varpi$ is ramified over the support of $\Gamma^{(m)}$, still no factor of 2 in $\varpi_m^*(\Gamma^{(m)})$, by our definition of $\Gamma^{(m)}$ as $1/2$ its support);

(ii)

(2.3.2) $$\varpi_{m*}(\Gamma_I[\alpha]) = (m-n)!\Gamma_{(n)}[\alpha] \quad, n = |I| > 1;$$

(iii)

(2.3.3) $$\varpi_{m*}(\Gamma_{I|\{i,j\}}) = \begin{cases} (m-n-2)!\Gamma_{(n|2)}, & n \neq 2; \\ 2(m-n-2)!\Gamma_{(2|2)}, & n = 2, \\ (1+\delta_{2,n})(m-n-2)!\Gamma_{(n|2)}, & \forall n \end{cases}$$

($\delta$= Kronecker delta);

(iv)

(2.3.4) $$\varpi_{m*}(\Gamma_{I\coprod\{i\}}) = (m-n-1)!\Gamma_{(n+1)};$$

(v)

(2.3.5) $$\varpi_{m*}(OF_j^{I:K-I|K^c-I}(\theta)) = a!b!F_j^{(n:a|b)}(\theta), \quad a = |K-I|, b = |K^c-I| = m-n-a;$$

moreover the number of distinct subsets $K-I$ with $a = |K-I|$, for fixed $I$ and $a$, is $\binom{m-n}{a}$.

Putting these together, we conclude

**Proposition 2.10.** *For any monoblock diagonal* $\Gamma_{(n)}, n > 1$*, we have an equivalence of codimension-1 cycles in* $\Gamma_{(n)}$*:*
(2.3.6)

$$\Gamma^{(m)}.\Gamma_{(n)} \sim \frac{1+\delta_{2,n}}{2}\Gamma_{(n|2)}+n\Gamma_{(n+1)}-\binom{n}{2}\Gamma_{(n)}[\omega]+\sum_s \frac{1}{\deg(\delta_s)}\sum_{a=0}^{m-n}\sum_{j=1}^{n-1}\beta_{n,j}\delta^n_{s,j*}F_j^{(n:a|m-n-a)}(\theta_s)$$

□

When $n = 2$, $\Gamma_{(n)}$ is just $2\Gamma^{(m)}$, hence

**Corollary 2.11.**

$$(\Gamma^{(m)})^2 \sim \frac{1}{2}\Gamma_{(2|2)} + \Gamma_{(3)} - \Gamma^{(m)}[\omega] + \sum_s \frac{1}{\deg(\delta_s)}\frac{1}{2}\sum_{a=0}^{m-2}\delta^2_{s,j*}F_1^{(2:1^a|1^{m-2-a})}(\theta_s). \tag{2.3.7}$$

□

**Corollary 2.12.** *We have*

(i)

$$\Gamma^{(m)}.\Gamma_{(2)}[\omega] = \Gamma_{(2|2)}[\omega] + 2\Gamma_{(3)}[\omega] - \Gamma_{(2)}[\omega^2] \tag{2.3.8}$$

(ii)
(2.3.9)

$$\Gamma^{(m)}.\Gamma_{(3)} = \frac{1}{2}\Gamma_{(3|2)}+3\Gamma_{(4)}-3\Gamma_{(3)}[\omega]+\sum_s \frac{3}{\deg(\delta_s)}\sum_{a=0}^{m-3}\delta^3_{s,1*}(F_1^{(3:1^a|1^{m-3-a})}(\theta_s)+\delta^3_{s,2*}F_2^{(3:1^a|1^{m-3-a})}(\theta_s)).$$

**Corollary 2.13.** *We have*

(i)

$$(\Gamma^{(2)})^k = \Gamma^{(2)}[(-\omega)^{k-1}] + \sum_s \frac{1}{\deg(\delta_s)}\frac{1}{2}\delta^2_{s,1*}(\Gamma^{(2)})^{k-2}.F_1^{(2:0|0)}(\theta_s)), k \geq 3;$$

$$\text{if } \dim(B) = 1, \int\limits_{X_B^{[2]}} \Gamma^{(2)})^3 = \frac{1}{2}\omega^2 - \frac{1}{2}\sigma, \qquad \sigma = |\{\textit{singular values}\}|;$$

(ii)

$$\begin{aligned}(\Gamma^{(3)})^3 &= -4\Gamma_{(3)}[\omega] + \Gamma^{(3)}[\omega^2]\\ &+\sum_s \frac{1}{\deg(\delta_s)}(3(\delta^3_{s,1*}F_1^{3:0|0}(\theta_s) + \delta^3_{s,2*}F_2^{3:0|0}(\theta_s)) + \frac{1}{2}\delta^2_{s,1*}\Gamma^{(3)}(F_1^{(2:1|0)}(\theta_s) + F_1^{(2:0|1)}(\theta_s)))\end{aligned}$$

[In Part (ii) we have used the elementary fact that $\omega.\theta_s = 0$, hence $\omega^i.F_1^{2:*}(\theta_s) = 0, \forall i > 0$, because this node scroll maps to $\theta_s$, more precisely to $2[\theta_s] \subset X_B^{(2)}$.]

**Corollary 2.14.** *If $m \le 3$ (resp. $m > 3$), then the class*

$$-\Gamma^{(m)}[\omega] + \sum_s \frac{1}{\deg(\delta_s)} \sum_{a=0}^{m-2} \delta^2_{s,1*} F_1^{(2:1^a|1^{m-2-a})}(\theta_s)$$

*(resp,*

$$\Gamma_{(2|2)} - \Gamma^{(m)}[\omega] + \sum_s \frac{1}{\deg(\delta_s)} \sum_{a=0}^{m-2} \delta^2_{s,1*} F_1^{(2:1^a|1^{m-2-a})}(\theta_s)$$

*is divisible by 2 in the integral Chow group of $X_B^{[m]}$.* □

To simplify notation we shall henceforth denote $\frac{1}{\deg(\delta_s)} \sum_s F^{\bullet}_{\bullet}(\theta_s)$ simply as $F^{\bullet}_{\bullet}$.

*Example* 2.15. This is presented here mainly as a check on some of the coefficients in the formulas above. For $X = \mathbb{P}^1$, $X^{(m)} = \mathbb{P}(H^0(\mathcal{O}_X(m))) = \mathbb{P}^m$, and the degree of $\Gamma^{(m)}_{(n)}$ is $n(m-n+1)$. Indeed this degree may be computed as the degree of the degeneracy locus of a generic map $n\mathcal{O}_X \to P^{n-1}_X(\mathcal{O}_X(m))$ where $P^k_X$ denotes the $k$-th principal parts or jet sheaf. It is not hard to show that $P^{n-1}_X(\mathcal{O}_X(m)) \simeq n\mathcal{O}_X(m-n+1)$.

For example, $\Gamma^{(3)}_{(2)}$ is a quartic scroll equal to the tangent developable of its cuspidal edge, i.e. the twisted cubic $\Gamma^{(3)}_{(3)}$. The rulings are the lines $L_p = \{2p+q : q \in X\}$, tangent to the $\Gamma^{(3)}_{(3)}$, each of which has class $-\frac{1}{2}\Gamma^{(3)}[\omega]$. Therefore by Corollary 2.11, the self-intersection of $\Gamma^{(3)}$ in $\mathbb{P}^3$ (or half the intersection of $\Gamma^{(3)}$ with $\Gamma^{(3)}_2$, as a class on $\Gamma^{(3)}_2$) is represented by $\Gamma^{(3)}_{(3)}$ plus one ruling $L_p$.

If $m = 4$ then $\Gamma^{(4)}$ is formally a cubic (half a sextic hypersurface) in $\mathbb{P}^4$, whose self-intersection, as given by Corollary 2.11, is half the Veronese $\Gamma_{(2|2)}$ plus the (sextic) tangent developable $\Gamma_{(3)}$, plus one osculating plane to the twisted quartic $\Gamma_{(4)}$, representing $-\Gamma^{(4)}[\omega]$ . □

Next we extend Proposition 2.10 to the polyblock case. Consider a distribution $\underline{n}$ or equivalently a shape $(n.) = (n.^{\mu.}), n_1 > ... > n_r$, and let $(I.)$ be any partition having this shape. Let $D^m_{(n^{\mu.}_.)}$ be the image of $\Gamma_{(n^{\mu.}_.)}$ in $X^{(m)}_B$. We identify

$$D^m_{(n^{\mu.}_.)} \simeq \prod_{j=1}^{r} {}_B(X_B^{(\mu_j))}) = \prod_{n=\infty}^{1} {}_B(X_B^{(\mu(n))}). \tag{2.3.10}$$

Generally, for any cohomology class $\alpha$ on $D^m_{(n^{\mu.}_.)}$, we will denote its pullback via the cycle map to $\Gamma_{(n^{\mu.}_.)}$ by $\Gamma_{(n^{\mu.}_.)}[\alpha]$. Some special cases of this are: for a collection $\alpha_1, ..., \alpha_r$ of cohomology classes on $X$, and $\forall\ \lambda_j \le \mu_j \forall j$, we will use the notation $\Gamma_{(n.^{\mu.})}[\alpha_1^{(\lambda_1)}, ..., \alpha_r^{(\lambda_r)}]$ as in Section 0.2.1; and more generally $\beta \star_t [\alpha.^{(\lambda.)}], t \le r$, for any (co)homology class $\beta$ on $\Gamma_{(n^{\mu.}_.)}$.

We need to set up analogous notations for the case of a node scroll and its base, which is a special diagonal locus associated with a boundary datum $(T, \delta, \theta)$. Thus fix such a boundary datum, and let $X', X"$ be the components of its normalization along $\theta$ (where by convention $X' = X"$ in case $\theta$ is nonseparating). Define *multidistribution data* $\phi$ of total length $n$ as

$$\phi = (n_\ell : \underline{n}'|\underline{n}")$$

where $(\underline{n}')\coprod(\underline{n}")$ is a distribution of total length $n - n_\ell$. In other words, if we set

$$\underline{n} = \underline{n}' \coprod \underline{n}" \coprod n_\ell,$$

then $\underline{n}$ is a distribution of length $n$. We will usually assume $(\underline{n})$ is full of length $n = m$. We view $\phi$ as obtained from $\underline{n}$ by removing a single block of size $n_\ell$ and declaring each remaining block as either $x$ type or $y$ type). Thus $\phi$ is the natural unordered analogue of a multipartition $\Phi$ and of course to each $\Phi = (J : I'.|I".)$ there is an associated mutlidistribution $\phi = (|J| : |I'.|\,|\,|I".)$.

The special diagonal locus corresponding to this multidistribution (and to the boundary datum) is of course

(2.3.11) $$X^\phi_\theta = \Gamma_{(n'.)}(X') \times_T \Gamma_{(n".)}(X") \subset (X')^{[n']}_T \times (X")^{[n"]}_T$$

where $\underline{n}' = (n'.^{(\mu.')}), \underline{n}" = (n".^{(\mu.")})$. It maps to $X^{(m)}_B$ by adding $n_\ell\theta$. If $\theta$ is nonseparating, so that $X' = X"$, we take $\underline{n}" = \emptyset$ as usual. The (unordered) *node scroll* $F^\phi_j(\theta)$ is the appropriate ($j$-th) component of the inverse image of $X^\phi_\theta$ in the Hilbert scheme. It is a $\mathbb{P}^1$-bundle over $X^\phi_\theta$, with bundle projection $c =$ restriction of cycle map.

Now generally, for any cohomology class $\alpha$ on $X^\phi_\theta$, we may define a class on $F^\phi_j(\theta)$ by

$$F^\phi_j(\theta)[\alpha] = c^*(\alpha).$$

More particularly, we will need the following type of class. First, set

(2.3.12) $$a(n_\ell : \underline{n}'|\underline{n}") = a(\underline{n}')a(\underline{n}").$$

Now for a collection of cohomology classes $\alpha'., \alpha".$ on $X', X".$ respectively, we define as in Section 0.2.1, an associated *twisted node scroll class* by

(2.3.13) $$F^\phi_j(\theta)[\alpha'.^{(\lambda.')}, \alpha".^{(\lambda.")}] = \frac{1}{a(\phi)}\varpi_{m*}(OF^\Phi_j(\theta)[\alpha.^{(\lambda.)}])$$

where $|\Phi| = \phi$ (cf. (2.2.21)). The numerical coefficient is just the reciprocal of the degree of the symmetrization map $\varpi : X^\Phi_\theta \to X^\phi_\theta$. Hence the twisted node scroll classes are just flat pullbacks of the analogous classes defined on the base $X^\phi$ of the node scroll $F^\phi_j(\theta)$, i.e.

$$F^\phi_j(\theta)[\alpha'.^{(\lambda.')}, \alpha".^{(\lambda.")}] = c^* X^\phi_\theta[\alpha'.^{(\lambda.')}, \alpha".^{(\lambda.")}].$$

Of course, we also set

$$F_j^{n_\ell/\underline{n}}(\theta)[\alpha.] = \sum_{(\underline{n}')\coprod(\underline{n}")=(\underline{n})\setminus n_\ell} F_j^{(n_\ell:\underline{n}'|\underline{n}")}(\theta)[\alpha.] \tag{2.3.14}$$

Also note that if $\underline{n}'\coprod\underline{n}" = \underline{n}\setminus\{n_\ell\}$, then we have

$$\frac{a(\underline{n}')a(\underline{n}")}{a(\underline{n})} = \frac{1}{\mu_{(\underline{n})}(n_\ell)}\frac{1}{\binom{\mu_{(\underline{n})}(n_\ell)-1}{\mu_{(\underline{n}')}(n_\ell)}} \tag{2.3.15}$$

and also

$$\frac{a(\underline{n}\setminus\{n_\ell\})}{a(\underline{n})} = \frac{1}{\mu(n(\ell))} \tag{2.3.16}$$

These are the respective ratios

$$\frac{\deg(OF_j^{(I_\ell:I'.|I".)} \to F_j^{(|I_\ell|:n'.|n".)})}{\deg(\Gamma_{(I.)} \to \Gamma_{(\underline{n})})}$$

when the special fibre is reducible or irreducible.

Similarly to the case of partitions, we will use the notation $u_{j,\ell}(n.^{\mu.})$to denote the new distribution (of the same total length) obtained from $(n.^{\mu.})$ by uniting a block of size $n_j$ with one of size $n_\ell$, i.e. the distribution whose frequency function $\mu_u$ coincides with $\mu$, except for the values

$$\begin{aligned}\mu_u(n_j+n_\ell) &= \mu(n_j+n_\ell)+1,\\ \mu_u(n_j) &= \mu(n_j)-1,\\ \mu_u(n_\ell) &= \mu(n_\ell)-1.\end{aligned}$$

There is an analogous operation on a cohomological vector $(\alpha(n)^{\lambda(n)})$ defined by

$$u_{j,\ell}(\alpha.^{\lambda.}) = \tag{2.3.17}$$
$$(...,\alpha(n_j+n_\ell)^{\lambda(n_j+n_\ell)}(\alpha(n_j)\cdot_X\alpha(n_\ell)),...,\alpha(n_j)^{\lambda(n_j)-1},...,\alpha(n_\ell)^{\lambda(n_\ell)-1},...,\alpha(1)^{\lambda(1)})$$

Now set

$$\begin{aligned}\nu_{(\underline{n})}(a,b) &= \frac{\mu_{(\underline{n})}(a+b)+1}{\mu_{(\underline{n})}(a)(\mu_{(\underline{n})}(b)-1)} \quad , \quad a=b\\ &= \frac{\mu_{(n.)}(a+b)+1}{\mu_{(\underline{n})}(a)\mu_{(\underline{n})}(b)} \quad , \quad a\neq b.\end{aligned} \tag{2.3.18}$$

Note that

$$\nu_{(\underline{n})}(n_j,n_\ell) = \frac{\deg(\Gamma_{U_{j,\ell}(I.)} \to \Gamma_{u_{j,\ell}(|I.|)})}{\deg(\Gamma_{(I.)} \to \Gamma_{(\underline{n})})} \tag{2.3.19}$$

Now the following result follows directly from Corollary 2.9 by adjusting for the degrees of the various symmetrization maps.

**Proposition 2.16.** *Let* $(n.) = (n_1^{\mu_1}|...|n_r^{\mu_r})$ *be a full distribution on* $[1, m]$, $n_1 > ... > n_r$, *and* $\alpha_1, ..., \alpha_r$ *cohomology classes on* $X$. *Let* $F^{\bullet}_{\bullet} = \sum_s F^{\bullet}_{\bullet}(\theta_s)$ *denote various weighted node scrolls, with reference to a fixed covering system of boundary data. Then we have, :*

$$\Gamma^{(m)}.\Gamma_{(n.)}[\alpha.^{\lambda.}.] \sim \sum_{j<\ell} \nu_{(n.)}(n_j, n_\ell) n_j n_\ell \Gamma_{(u_{j,\ell}(n.))}[u_{j,\ell}(\alpha.^{\lambda.})]$$
$$- \sum_\ell \Gamma_{(n)}[\alpha.] \star_1 [\binom{n_1}{2}\omega, ..., \binom{n_r}{2}\omega]$$
$$+ \sum_{\theta_s \text{ separating}} \sum_\ell \sum_{n'.\coprod n".=n.\setminus\{n_\ell\}} \frac{1}{\mu(n_\ell)} \frac{1}{\binom{\mu(n_\ell)-1}{\mu_{(n'.)}(n_\ell)}} \sum_{j=1}^{n_\ell-1} \beta_{n_\ell,j} F_j^{(n_\ell:n'.|n".)(\theta_s)}[\alpha.]$$
$$+ \sum_{\theta_s \text{ nonseparating}} \sum_\ell \frac{1}{\mu(n_\ell)} \sum_{j=1}^{n_\ell-1} \beta_{n_\ell,j} F_j^{(n_\ell:n'.|n".)}(\theta)[\alpha.]$$

□

*Example* 2.17. We have

$$\Gamma^{(m)}.\Gamma_{(2|2)} \sim \tag{2.3.20}$$
$$\frac{3}{2}\Gamma_{(2|2|2)} + 2\Gamma_{(4)} + 2\Gamma_{(3|2)} - \Gamma_{(2|2)} \star_1 [\omega] + \sum_{a=0}^{m-4} \frac{1}{2\binom{m-4}{a}} (F_1^{2:2,1^a|1^{m-4-a}} + F_1^{2:1^a|2,1^{m-4-a}})$$

**Corollary 2.18.** *We have*

$$(\Gamma^{(m)})^3 \sim \tag{2.3.21}$$
$$\frac{3}{4}\Gamma_{(2|2|2)} + 4\Gamma_{(4)} + \frac{3}{2}\Gamma_{(3|2)} - \Gamma_{(2|2)} \star_1 [\omega] - 4\Gamma_{(3)}[\omega] + \Gamma^{(m)}[\omega^2]$$
$$+ \frac{1}{2} \sum_{a=0}^{m-4} \frac{1}{2\binom{m-4}{a}} (F_1^{2:2,1^a|1^{m-4-a}} + F_1^{2:1^a|2,1^{m-4-a}})$$
$$+ 3 \sum_{a=0}^{m-3} (F_1^{(3:1^a|1^{m-3-a})} + F_2^{(3:1^a|1^{m-3-a})})$$
$$+ \frac{1}{2} \sum_{a=0}^{m-2} \Gamma^{(m)}.F_1^{(2:1^a|1^{m-2-a})}$$

Proposition 2.16 completes one major step in the proof of Theorem 2.1. The remaining step will be taken in the next subsection.

2.4. **Polarized node scrolls.** What now remains to be done to complete the proof of Theorem 2.1 is to work out the intersection product of the discriminant polarization $-\Gamma^{(m)}$, and all its powers, with a node scroll $F$. As discussed in the

introduction to this chapter, this will be accomplished via an analysis of the polarized structure of a node scroll, refining the one given in Lemma 1.10. There we found one (decomposable) vector bundle $E_0$ (over a suitable special diagonal locus), such that $F = \mathbb{P}(E_0)$. The point now is effectively to find a line bundle $L$ such that the twisted vector bundle $E = E_0 \otimes L$ – which also has $F = \mathbb{P}(E)$) – is the 'right one' in the sense that the associated polarization $\mathcal{O}_{\mathbb{P}(E)}(1)$ coincides with the restriction of the discriminant polarization, i.e. $\mathcal{O}_F(-\Gamma^{(m)})$. We will focus first on 'maximal' node scrolls (ones with no diagonal conditions); the formula thus obtained is of course still valid in the nonmaximal case but there is can and will be usefully explicated. Finally, we will consider everything first in the ordered case and descend to the unordered one at the end of the section.

2.4.1. *Pencils.* We begin by studying the case where $X/B$ is a 1-parameter family of curves with a finite number of singular members, all 1-nodal (the '1-parameter, 1-nodal' case). While in this case specifying a singular fibre is equivalent to specifying a node, in the general, higher-dimensional base case, it is the latter that carries through. As in our study of diagonal classes, we will first consider the ordered case. To begin with, we review and amplify some notations relating to (multi)partitions and associated diagonal loci for a *special* fibre $X_s$, analogous to those established previously in the relative case over $B$. Thus, fix a singular fibre $X_s$ with unique node $\theta = \theta_s$, and let $(X', \theta_x), (X", \theta_y)$ be the components of its normalization with the distinct node preimages marked; if $X_s$ is irreducible (or equivalently, $\theta$ is nonseparating), then $X' = X"$ as global varieties, but they differ in the marking; if necessary to specify the singular value $s$, the same will be denoted $(X'_s, \theta_{x,s}), (X"_s, \theta_{y,s})$ .

In this setting, we recall that a *(full) set of multipartition data*

$$\Phi = (J : I'.|I".)$$

consists of a 'nodebound' block $J = 0(\Phi)$, plus a pair of '$x$ and $y$' partitions $I' = x(\Phi), I" = y(\Phi)$, such that $(I.) = J \coprod (I'.) \coprod (I".)$ is a full partition on $[1, m]$. Set

$$n = |J|, I' = \bigcup_{\ell} I'_{\ell}, I" = \bigcup_{\ell} I"_{\ell}, n' = |I'|, n" = |I"|,$$

and let

$$\underline{n}' = (n'.^{\mu'.}) := (|I'.|), \underline{n}" = (n".^{\mu".}) := (|I".|)$$

be the associated distributions and shapes (with $n'_1 > n'_2 > ...$ and ditto $n"$.). The *multidistribution* associated to $\Phi$ is by definition

$$\phi = |\Phi| = (n : \underline{n}'.|\underline{n}".).$$

Here again $n, \underline{n}', \underline{n}"$ are referred to as the *nodebound*, $x$-, and $y$- portions of $\phi$, and denoted $\theta(\phi), x(\phi), y(\phi)$ respectively.

We will say that $\Phi$ is *maximal* if each $I'_{\ell}$ and $I"_{\ell}$ is a singleton (and thus represents a vacuous condition). In general, we will say that $\Phi_1 \prec \Phi_2$, where $\Phi_1, \Phi_2$ are

full multipartition data, if they have the same $J$ block, and if $(I'_{1.}) \prec (I'_{2.}), (I"_{1.}) \prec (I"_{2.})$ in the sense defined earlier (i.e. if each non-singleton block of $\Phi_2$ is contained in a block of $\Phi_1$).

As before, we define

$$X_s^\Phi = X^\Phi = (X')^{(I'.)} \times (X")^{(I".)}.$$

The notation means that the coordinates in a given block are set equal to each other. Therefore

$$X^\Phi \simeq X^{r'} \times X^{r"},$$

where $r', r"$ are the respective numbers of blocks in $(I'.), (I"_.)$. If $X_s$ is irreducible, $X_s^\Phi$ depends only on $(I'.) \coprod (I".)$, therefore in a global context we may, and will in this case, always take $I" = \emptyset$; however in a local context, specifying $(I'.), (I".)$ specifies a sheet of $X_s^\Phi$ over the origin. We have

$$\Phi_1 \prec \Phi_2 \Rightarrow X^{\Phi_1} \subset X^{\Phi_2}, \tag{2.4.1}$$

an embedding of smooth varieties. As before, $X_s^\Phi$ maps to the Cartesian product $X_s^m$ by putting $0$ in the $J$ coordinates. The image is defined by the following equations:

- horizontal

$$y_i = 0, \forall i \in J \cup I',\ I' := \bigcup_\ell I_\ell\ ; \tag{2.4.2}$$

- vertical

$$x_i = 0, \forall i \in J \cup I",\ I" := \bigcup_\ell I"_\ell\ ; \tag{2.4.3}$$

- diagonal

$$x_{i_1} - x_{i_2} = 0, \forall i_1, i_2 \in I'_\ell, \forall \ell; y_{i_1} - y_{i_2} = 0, \forall i_1, i_2 \in I"_\ell, \forall \ell. \tag{2.4.4}$$

Notice that the 'origins' $\theta_x, \theta_y$ (i.e. node preimages) induce a stratification of $X^\Phi$, where the stratum $\mathcal{S}_{k',k"}$ of codimension $k = k' + k"$ is the locally closed locus of points having exactly $k'$ (resp. $k"$) of their components indexed by $I'$ (resp. $I"$) equal to the origin $\theta_x$ (resp. $\theta_y$).

The node scroll $OF_j^\Phi = OF_j^\Phi(\theta)$ is a component of the inverse image $oc_m^{-1}(X^\Phi)$. It is a $\mathbb{P}^1$ bundle over $X^\Phi$ whose fibre may be identified with $C_j^n$ over the 0-stratum $\mathcal{S}_{0,0}$. There, in terms of the local model $H_n$, the fibre $C_j^n \subset \mathbb{P}^{n-1}$ is defined by the homogeneous equations

$$Z_1 = ... = Z_{j-1} = Z_{j+2} = ... = Z_n = 0.$$

In other words, homogeneous coordinates on $C_j^n \sim \mathbb{P}^1$ are given by $Z_j, Z_{j+1}$. Now recall that under our identification of the local model $H_n$ with the blowup of the discriminant, the $Z_i$ correspond to the generators $G_{i,J}$ given by the mixed Van der Monde determinants (1.7.3) in the $J$-indexed variables.

Similarly, in terms of the local model $H_m$ over a neighborhood of the 'origin' $(\theta_x)^{I'} \times (\theta_y)^{I"}$, i.e. the smallest stratum $\mathcal{S}_{n'+n"}$, the fibre $C^m_{j+n"}$ is coordinatized by $Z_{j+n"}, Z_{j+n"+1}$, with the other $Z$ coordinates vanishing, and these correspond to the generators $G_{j+n"}, G_{j+n"+1}$ (in all the variables).

Likewise, over a neighborhood of a point $p$ in the stratum $\mathcal{S}_{k',k"} \subset X^\Phi$, we have a local model $H_{n+k'+k"}$ for the Hilbert scheme, and there the fibre becomes $C^{n+k'+k"}_{j+k"}$ and is coordinatized by $Z_{j+k"}, Z_{j+k"+1}$, which correspond to the generators $G_{j+k"}, G_{j+k"+1}$ in the appropriate variables (where the components of $p$ are equal to $\theta_x$ or $\theta_y$).

We need to analyze the mixed Van der Monde determinants restricted on $X^\Phi$. To this end, assume to begin that $\Phi$ is maximal in that $(I'.), (I".)$ have *singleton* blocks. We also assume for now that the singular fibre $X_s$ in question is reducible. We work in the local model $H_n$ over a neighborhood of the origin. Consider a Laplace expansion, along the $J$-indexed columns, of the mixed Van der Monde determinant that yields $G_{j+n"}$. This expansion has one $n \times n$ subdeterminant that is equal to $G_{j,J}$, and in particular is *constant* along $X^\Phi$; the complementary submatrix to this, restricted on the locus $X^\Phi$, itself splits in two blocks, of size $n' \times n', n" \times n"$, which are themselves 'shifts' of ordinary Van der Mondes, in the $x_{I'}, y_{I"}$ variables respectivley, where the exponents are shifted up by $n-j+1$ (resp. $j-1$). The determinant of this complementary matrix on $X^\Phi$ equals, using block expansion,

$$\gamma_j^{(n:I'|I")} = (x^{I'})^{n-j+1} (y^{I"})^{j-1} \prod_{a<b\in I'} (x_a - x_b) \prod_{a<b\in I"} (y_a - y_b) \tag{2.4.5}$$

(where $x^{I'} = \prod\limits_{i\in I'} x_i$ etc.). Note that $x^{I'}$ is a defining equation for the '$x$-boundary'

$$\partial_x X^\Phi = \bigcup_{i\in I'} X^{\Phi\setminus i} =: X^{\partial_x \Phi} \tag{2.4.6}$$

where $\Phi \setminus i$ means remove $i$ from $I'$ and add it to $J$ and map the appropriate locus to $X^\Phi$ as a Cartier divisor by putting $\theta_x$ at the $i$th coordinate; similarly $y^{I"}$. The other factors of $\gamma_j^{(n:I'|I")}$ define respectively the big $x$ diagonal $D^\Phi_x$ and big $y$ diagonal $D^\Phi_y$ on $X^\Phi$, i.e.

$$D^\Phi_x = p^*_{(X')^{I'}}(D^{I'}_{X'}), D^\Phi_y = p^*_{(X")^{I"}}(D^{I"}_{X"}) \tag{2.4.7}$$

where $D_{X'}.D_{X"}$ are the usual big diagonals. Note also that the $G_{j,J}$ are globally defined along $X^\Phi$. Now define a line bundle on $X^\Phi$ as follows:

$$OE^\Phi_{s,j} = \mathcal{O}_{X^\Phi}(-(n-j+1)\,\partial_x(X^\Phi) - (j-1)\,\partial_y(X^\Phi) - D^\Phi_x - D^\Phi_y), \tag{2.4.8}$$

($s$ will be omitted when understood).

In the irreducible case there is no distinction globally between $I'$ and $I"$, so we may as well assume $I" = \emptyset$ and take $D^\Phi_y = 0$; the $\partial_x$ and $\partial_y$ are still defined, and

different, based on setting the appropriate coordinates equal to $\theta_x$ or $\theta_y$. With this understood, we still define $OE^{\Phi}_{s,j}$ as in (2.4.8). Then the foregoing calculations have the following conclusion (NB we are using the quotient convention for projective bundles).

**Proposition 2.19.** *Let $X_s$ be a singular fibre and $\Phi = (J : I'.|I".)$ a set of multipartition data where $I". = \emptyset$ if $X_s$ is irreducible. Define line bundles $OE^{\Phi}_{s,j}$ on $X^{\Phi}_s$ by*

$$OE^{\Phi}_{s,j} = OE^{\Phi_{\max}}_{s,j} \otimes \mathcal{O}_{X^{\Phi}} \tag{2.4.9}$$

*where $\Phi_{\max}$ is the unique maximal partition dominating $\Phi$ and $OE^{\Phi_{\max}}_{s,j}$ is defined by (2.4.8). Then we have an isomorphism of $\mathbb{P}^1$-bundles over $X^{\Phi}$*

$$OF^{\Phi}_{s,j} \simeq \mathbb{P}(OE^{\Phi}_{s,j} \oplus OE^{\Phi}_{s,j+1}) \tag{2.4.10}$$

*that induces an isomorphism*

$$\mathcal{O}(-\Gamma^{\lceil m \rceil}) \otimes \mathcal{O}_{OF^{\Phi}_s} \simeq \mathcal{O}_{\mathbb{P}(OE^{\Phi}_{s,j} \oplus OE^{\Phi}_{s,j+1})}(1); \tag{2.4.11}$$

*these isomorphisms are uniquely determined by the condition that over a neighborhood of a point in $\mathcal{S}_{k',k"} \subset X^{\Phi}_s$, they take the generators $G_{j+k"}, G_{j+k"+1}$ to generators $\gamma^{(n:I'|I")}_j, \gamma^{(n:I'|I")}_j$ of $OE^{\Phi}_{s,j}, OE^{\Phi}_{s,j+1}$, respectively.*

*Proof.* To begin with, note that the requirement of compatibility of (2.4.10) with (2.4.24) determines it uniquely over any open set of $X^{\Phi}_s$.

Next, note that every multipartition set $\Phi$ is dominated by a maximal one and the appropriate $\mathbb{P}^1$ bundles and polarizations restrict in the natural way. In fact, quite generally, whenever $\Phi_1 \prec \Phi_2$ are multipartitions, we have a Cartesian diagram of polarized $\mathbb{P}^1$-bundles:

$$\begin{array}{ccc} OF^{\Phi_1}_j & \to & OF^{\Phi_2}_j \\ \downarrow & \square & \downarrow \\ X^{\Phi_1} & \to & X^{\Phi_2}. \end{array} \tag{2.4.12}$$

Therefore is suffices to prove the assertions in case $\Phi$ is maximal. In that case the foregoing discussion yields the claimed isomorphisms in a neighborhood of the 0-stratum $\mathcal{S}_{0,0}$. A similar argument applies in a neighborhood of a point in any other stratum. The compatibility of all these isomorphisms with (2.4.24) ensures that the local isomorphisms glue together to a global one.

□

Note that (2.4.10) reproves Lemma 1.10, though the latter, of course, does not yield the 'correct' polarization and is therefore of little use enumeratively.

*Example* 2.20. We have

$$OF^{(12:3|\emptyset)}_{s,1} = \mathbb{P}_{X'}(\mathcal{O}(-2\theta_x) \oplus \mathcal{O}(-\theta_x)) \tag{2.4.13}$$

Consequently

$$(-\Gamma^{(3)})^2.OF_{s,1}^{(12:3|\emptyset)} = -3. \tag{2.4.14}$$

Of course, in this example ordering is irrelevant.

2.4.2. *General families: maximal multipartition.* We now take up the extension of Proposition 2.19 to the setting of an arbitrary nodal family $X/B$, where a maximal node scroll $F_j^n(\theta)$ is associated to a relative node $\theta$, or more precisely to a boundary datum $(T, \delta, \theta)$, as in §1.5 and §1.8. In this setting the scrolls $F_j^n(\theta)$ are (polarized, via the discriminant) $\mathbb{P}^1$-bundles over $(X_T^\theta)^{[m-n]}$ defined for all $1 \leq j < n \leq m$, and we aim to identify them, or rather more conveniently, their pullbacks over the ordered Hilbert scheme $(X_T^\theta)^{\lceil m-n\rceil}$. As in the foregoing discussion in the 1-parameter case, the polarized $\mathbb{P}^1$-bundle $F_j^n(\theta)$ is just the projectivization of the rank-2 bundle that is the direct sum of the invertible ideals of the intermediate diagonals $OD_j^m(\theta), OD_{j+1}^m(\theta)$ via the map 'add $n\theta$'. Now these ideals were determined in Proposition 1.22. Therefore we conclude

**Proposition 2.21.** *Let $X/B$ be a family of nodal curves and $(T, \delta, \theta)$ be a boundary datum as in §1.5. Let $OE_j^n(\theta)$ be the line bundle over $(X_T^\theta)^{\lceil m-n\rceil}$ defined (in divisor notation) by*

$$\begin{aligned} OE_j^n(\theta) = &-(n-j+1)(\theta_x)^{\lceil m-n\rceil} - (j-1)(\theta_y)^{\lceil m-n\rceil} - \Gamma^{\lceil m-n\rceil} \\ &+ ((\pi^\theta)^{\lceil m-n\rceil})^*(\binom{n-j+1}{2}\psi_x + \binom{j}{2}\psi_y). \end{aligned} \tag{2.4.15}$$

*Then the pullback of the node scroll $F_j^n(\theta)$ on $(X_T^\theta)^{\lceil m-n\rceil}$ is polarized-isomorphic to $\mathbb{P}(OE_j^n(\theta) \oplus OE_{j+1}^n(\theta))$.*

*Remark* 2.22. Note that interchanging the $x$ and $y$ branches along $\theta$ interchanges $OE_j^n(\theta)$ and $OE_{n-j+1}^n(\theta)$, hence also the node scrolls $F_j^n$ and $F_{n-j}^n(\theta)$.

2.4.3. *General families: nonmaximal multipartition.* Next we extend the basic formula (2.4.15) for the line bundles $OE_j^\Phi(\theta)$ making up the node scroll to the case where $\Phi$ is a general, not necessarily maximal, multi-partition. This is essentially a matter of computing the restrictions of the various 'constituents' of $OE_j^\Phi(\theta)$ on a general diagonal locus, and is readily done based on our earlier results, notably Proposition 2.9.

Thus let $\Phi = (J : I'.|I".)$ be a full multipartition on $[1, m]$ where as usual we take $I". = \emptyset$ if $\theta$ is a nonseparating node. We set $n = |J|, k = m - n$ where we may assume $k \leq m - 2$, and let

$$OD_\Phi^\theta \subset (X_T^\theta)^k, \Gamma_\Phi^\theta \subset (X_T^\theta)^{\lceil k\rceil}$$

be the associated diagonal loci. Note that these are ordinary diagonal loci associated to the family $X_T^\theta$ (which is disconnected when $\theta$ is separating); in the

nonseparating case, $\Gamma^\theta_\Phi = \Gamma_{(I'.)}$ (on $X^\theta_T$). Therefore, to start with, the restriction of $\Gamma^{\lceil k\rceil}$ on $\Gamma^\theta_\Phi$ is computed by Proposition 2.9.

As for $\theta_x, \theta_y$, the restriction is quite elementary (and comes from the corresponding transversal intersection on the cartesian product); namely, in the reducible case,

$$\theta_x^{\lceil k\rceil}.\Gamma^\theta_\Phi = \sum_\ell |I'_\ell| p_{\min(I'_\ell)}(\theta_x)$$

$$\theta_y^{\lceil k\rceil}.\Gamma^\theta_\Phi = \begin{cases} \sum\limits_\ell |I"_\ell| p_{\min(I"_\ell)}(\theta_y), & \theta \text{ separating} \\ \sum\limits_\ell |I'_\ell| p_{\min(I'_\ell)}(\theta_y), & \theta \text{ nonseparating} \end{cases} \tag{2.4.16}$$

Summarizing, we have

**Corollary 2.23.** *In the situation above we fix an arbitrary full multipartition $\Phi$ and boundary datum $(T, \delta, \theta)$ and identify a divisor class with the corresponding line bundle. Then we have on $\Gamma_\Phi = \Gamma^\theta_\Phi$, suppressing the node $\theta$ for brevity:*
*if $\theta$ is separating,*

$$\begin{aligned} OE^\Phi_j \sim & -\Gamma^{\lceil k\rceil}.\Gamma_\Phi + \sum_\ell p^*_{\min(I'_\ell)}(-|I'_\ell|(n-j+1)\theta_x)) \\ & + \sum_\ell p^*_{\min(I"_\ell)}(-|I"_\ell|(j-1)\theta_y) + ((\pi^\theta)^{\lceil m-n\rceil})^*(\binom{n-j+1}{2}\psi_x + \binom{j}{2}\psi_y).; \end{aligned} \tag{2.4.17}$$

*if $\theta$ is nonseparating,*

$$\begin{aligned} OE^\Phi_j \sim & -\Gamma^{\lceil k\rceil}.\Gamma_\Phi + \sum_\ell p^*_{\min(I'_\ell)}(-|I'_\ell|((n-j+1)\theta_x + (j-1)\theta_y) \\ & + ((\pi^\theta)^{\lceil m-n\rceil})^*(\binom{n-j+1}{2}\psi_x + \binom{j}{2}\psi_y). \end{aligned} \tag{2.4.18}$$

2.4.4. *Unordered cases.* Finally we carry our results over to the unordered case, i.e. node scrolls over diagonal loci in the Hilbert scheme itself. This is straightforward, as both the scrolls and related line bundles descend. To state the result, we use the following notation: let

$$\phi = (n : \underline{n}'|\underline{n}") \tag{2.4.19}$$

be a full multidistribution, i.e. a natural number plus 2 distributions such that the total length

$$n + \sum_\ell \underline{n}'(\ell) + \sum_\ell \underline{n}"(\ell) = m.$$

The shape of $\phi$, $(n : (n'.^{\mu'}\cdot)|(n".^{\mu"}\cdot))$ is defined as before. To a multipartition $\Phi$ as above we associate the multidistribution

$$\phi = |\Phi| = (|J| : |I'.||I".|).$$

The locus $\Gamma^\theta_\phi \subset (X^\theta_T)^{[k]}$, $k \leq m-2$, and over it, the scrolls $F^\phi_j(\theta)$ are defined as before. As before, we let

$$\varpi_\Phi : \Gamma^\theta_\Phi \to \Gamma^\theta_\phi$$

be the natural symmetrization map, of degree $a(n'.)a(n".)$. We also define in the separating case, for collections

$$\alpha'_1, ..., \alpha'_{r'}, \alpha"_1, ..., \alpha"_{r"}$$

of cohomology classes on $X$ ('twisting classes'),

$$\Gamma^\theta_\phi \star_k [\alpha'.; \alpha".] = s_k((X')^{(\mu'_1)}[\alpha'_1], ..., (X")^{(\mu"_1)}[\alpha"_1], ...) \tag{2.4.20}$$

where $s_k$ is the $k$th elementary symmetric function (in all the $r' + r"$ indicated variables). There is an analogous notion, with a single collection of twisting classes, in the nonseparating case.

Then the appropriate line bundles on $\Gamma_\phi$ are here given up to numerical equivalence by:

$\theta$ <u>separating</u>:

$$\begin{aligned} E^\phi_j(\theta) \sim_{\text{num}} \Gamma^{(k)}.\Gamma^\theta_\phi + \Gamma^\theta_\phi \star_1 [-n'.(n-j+1)\theta_x \,; -n".(j-1)\theta_y] \\ + ((\pi^\theta)^{(m-n)})^*(\binom{n-j+1}{2}\psi_x + \binom{j}{2}\psi_y); \end{aligned} \tag{2.4.21}$$

$\theta$ <u>nonseparating</u>:

$$\begin{aligned} E^\phi_j(\theta) \sim_{\text{num}} \Gamma^{(k)}.\Gamma^\theta_\phi + \Gamma^\theta_\phi \star_1 [-n'.(n-j+1)\theta_x \ ] \\ + ((\pi^\theta)^{(m-n)})^*(\binom{n-j+1}{2}\psi_x + \binom{j}{2}\psi_y); \end{aligned} \tag{2.4.22}$$

These bundles have the property, easy to check, that they pull back to $OE_{s,j}$ whenever $\phi$ is the multidistribution associated to $\Phi$. This suffices to ensure they are the 'correct' bundles at least up to torsion. Thus,

**Proposition 2.24.** *For each boundary datum $(T, \delta, \theta)$ and multidistribution $\phi$, we have a polarized isomorphism of $\mathbb{P}^1$-bundles over $(X^\theta_T)^{[k]}$*

$$F^\phi_j(\theta) \simeq \mathbb{P}(E^\phi_j(\theta) \oplus E^\phi_{j+1}(\theta)) \tag{2.4.23}$$

*where polarized means it induces an isomorphism*

$$\mathcal{O}(-\Gamma^{(m)}).F^\phi_j\theta) \simeq \mathcal{O}_{\mathbb{P}(E^\phi_j(\theta)\oplus E^\phi_{j+1}(\theta))}(1). \tag{2.4.24}$$

*Proof.* To begin with, in case $\phi$ is maximal the result follows from the ordered case by (faithful) flatness of the symmetrization map.

Given this, one is reduced to checking that the pullback $E^{\phi_{\max}}_{s,j}$ to $\Gamma_\phi$ is as in (2.4.21) This can be done as in the case of relative diagonal loci (see Proposition 2.16) ☐

To state the next result compactly, we introduce the following formal polynomial

$$s_k(a,b) = a^k + a^{k-1}b + \dots + b^k = (a^{k+1} - b^{k+1})/(a-b). \tag{2.4.25}$$

Thus $s_0 = 1, s_1 = a+b$ etc. Also, to avoid confusion, we recall that the polarization on a node scroll is given by $-\Gamma^{(m)}$ rather than $+\Gamma^{(m)}$.

**Corollary 2.25.** *For a node scroll $F = F_j^\phi(\theta)$ and all $\ell \geq 2$, we have, setting $e_j = E_j^\phi(\theta)$:*

$$(-\Gamma^{(m)})^\ell|_F = s_{\ell-1}(e_j, e_{j+1})(-\Gamma^{(m)}) - e_j e_{j+1} s_{\ell-2}(e_j, e_{j+1}). \tag{2.4.26}$$

*In particular, for any twist $\alpha.$, $(-\Gamma^{(m)})^\ell.F[\alpha.]$ is a linear combination of twisted node scrolls and twisted node sections.*

*Proof.* It follows from a standard, analogous formula for the self-intersection of the polarization on the projectivization of an arbitrary decomposable bundle, which can be easily proved by induction, starting from Grothendieck's formula for the case $k = 2$. □

*Remark* 2.26. Note that in our case, $e_j - e_{j+1} = \theta_x - \theta_y$, so the latter formula can be rewritten as

$$(-\Gamma^{(m)})^\ell|_F = ((e_j^\ell - e_{j+1}^\ell)(-\Gamma^{(m)}) - e_j e_{j+1}(e_j^{\ell-1} - e_{j+1}^{\ell-1}))/([\theta_x - \theta_y]). \tag{2.4.27}$$

The dividing by $[\theta_x - \theta_y]$ must be done with care, i.e. one must first expand the numerator in some ring where $[\theta_x - \theta_y]$ is not a zero divisor, then do the dividing, and only then impose the remaining relations defining the Chow ring of $F$.

*Remark* 2.27. It is worth noting that intersection products involving the sections $\theta_x, \theta_y$ are 'elementary' in view of the fact that

$$\theta_x^j = (-\omega)^{j-1}.\theta_x, j \geq 1. \tag{2.4.28}$$

Also, at least on the cartesian product $(X_T^\theta)^k$, $p_i^*(\theta_x)$ is geometrically a copy of $(X_T^\theta)^{k-1}$ embedded via inserting $\theta_x$ at the $i$-th coordinate, and similarly for products $p_i^*(\theta_x)p_j^*(\theta_x), i \neq j$ etc. Also, $\theta_x\theta_y = 0$, as the sections are disjoint. On the other hand intersections on $T$ involving the $\psi$ classes ultimately reduce to pure psi products, which are the subject of the Witten conjecture, as proven by Kontsevich [4].

Note that in the 'extreme' case $m = n$, the $E_j^\phi(\theta)$ and the node scroll $F_j^\phi(\theta)$ live on the base itself $T$ of the boundary datum and we have

$$E_j^\phi(\theta) = \binom{m-j+1}{2}\psi_x + \binom{j}{2}\psi_y. \tag{2.4.29}$$

*Example* 2.28. For $m = n = 2, F = F_1^2(\theta)$, we have

(2.4.30)
$$(-\Gamma^{(2)})^k|_F = (\psi_x^{k-1} + \psi_x^{k-2}\psi_y + ... + \psi_y^{k-1})(-\Gamma^{(2)}) - \psi_x\psi_y(\psi_x^{k-2} + \psi_x^{k-3}\psi_y + ... + \psi_y^{k-2}).$$

In particular, for $k = \dim(B) = \dim(F) = 1 + \dim(T)$, we have

(2.4.31)
$$(-\Gamma^{(2)})^k.F = \int\limits_T (\psi_x^{k-1} + \psi_x^{k-2}\psi_y + ... + \psi_y^{k-1}).$$

Note that if $B = \overline{\mathcal{M}}_g$ and $T = \overline{\mathcal{M}}_{i,1} \times \overline{\mathcal{M}}_{g-i,1}$, $1 \leq i \leq g/2$ (the usual $i$-th boundary component), the latter integral reduces to

$$\int\limits_{\overline{\mathcal{M}}_i} \psi_x^{3i-3} \int\limits_{\overline{\mathcal{M}}_{g-i}} \psi_y^{3(g-i)-3}$$

Note that (2.4.30) and (2.3.10) together imply

**Corollary 2.29.** *(i) The powers of the polarization on $X_B^{[2]}$ are*

(2.4.32)
$$(-\Gamma^{(2)})^k = -\Gamma[\omega^{k-1}] +$$
$$\frac{1}{2}\sum_s \delta_{s*}((\psi_x^{k-3} + \psi_x^{k-4}\psi_y + ... + \psi_y^{k-3})(-\Gamma^{(2)}) - \psi_x\psi_y(\psi_x^{k-4} + \psi_x^{k-5}\psi_y + ... + \psi_y^{k-4}))$$

*(ii) The image of the latter class on the symmetric product $X_B^{(2)}$ equals*

(2.4.33)
$$-\Gamma[\omega^{k-1}] + \frac{1}{2}\sum_s \delta_{s*}((\psi_x^{k-3} + \psi_x^{k-4}\psi_y + ... + \psi_y^{k-3})$$

*Proof.* (2.4.32) has been proved above; (2.4.33) follows because in the last summation in (2.4.32), the terms without $\Gamma^{(2)}$, i.e. the twisted node scroll, collapses under the cycle map to $X_B^{(2)}$. $\square$

*Example* 2.30. $m = 3, n = 2, \dim(B) = 1$:

(2.4.34)
$$(-\Gamma^{(3)})^2.F_1^{(2:1|0)}(\delta) = -3$$

(see Example 2.20). Consequently, in view of Corollary 2.13, we conclude

(2.4.35)
$$\int\limits_{X_B^{[4]}} (\Gamma^{(3)})^4 = 13\omega^2 - 9\sigma$$

(recall that each $F_i^{(3,0|0)}, i = 1, 2$ is a line with respect to the discriminant polarization $-\Gamma^{(3)}$.

2.5. **Tautological module.** We are now in position to give the formal definition of the tautological module $T^m$ and the proof of Theorem 2.1.

**Definition 2.31.** *Given a cohomology theory $A^\cdot$ and a $\mathbb{Q}$-subalgebra $R \subset A^\cdot(X)_\mathbb{Q}$ containing the canonical class $\omega$, the tautological module $T^m_R$ is the $R$-submodule of $A^\cdot(X^{[m]}_B)$ generated by the twisted diagonal classes $\Gamma_{(n.\mu.)}[\alpha.]$ and the direct images on $X^{[m]}_B$ the twisted node scroll classes $F^\phi_j(\theta)[\alpha.]$ and the twisted node scroll sections $-\Gamma^{(m)}.F^\phi_j(\theta)[\alpha]$ as $(T, \delta, \theta)$ ranges over a fixed covering system of boundary data for the family $X/B$. For the default choice $R = \mathbb{Q}[\omega]$, we denote $T^m_R$ by $T^m$.*

*Proof of Theorem 2.1.* . We wish to compute the product of a tautological class $c$ by $\Gamma^{(m)}$. If $c$ is a (twisted) diagonal class $\Gamma_{(n.\mu.)}[\alpha.]$, this is clear from Proposition 2.16. If $c$ is a twisted node scroll class $F^\phi_{s,j}$, it is obvious. Finally if $c$ is a node scroll section $-\Gamma^{(m)}.F^\phi_{s,j}$ it is clear from the case $k = 2$ of Corollary 2.25. □

*Remark* 2.32. In the important special case of computing a power $(\Gamma^{(m)})^k$ it is probably more efficient not to proceed by simple recursion, but rather to apply just Proposition 2.16 repeatedly to express $(\Gamma^{(m)})^k$ in terms of twisted diagonals plus classes $(\Gamma^{(m)})^t.F$ for various $t$'s and various $F$'s; then each of the latter classes can be computed at once using Corollary 2.25.

## 3. Tautological transfer and Chern numbers

In this chapter we will complete the development of our intersection calculus. First we study the *transfer* operation $\tau_m$, taking cycles on $X^{[m-1]}_B$ to cycles (of dimension 1 larger) on $X^{[m]}_B$, via the flaglet Hilbert scheme $X^{[m,m-1]}_B$. In the Transfer Theorem 3.3 we will show in fact that for any tautological class $u$ on $X^{[m-1]}_B$, the image $\tau_m(u)$ is a simple linear combination of basic tautological classes on $X^{[m]}_B$. We then review a splitting principle established in [12], which expresses the Chern classes of the tautological bundle $\Lambda_m(E)$, pulled back on $X^{[m,m-1]}_B$, in terms of those of $\Lambda_{m-1}(E)$, the discriminant polarization $\Gamma^{(m)}$, and base classes. Putting this result together with the Module Theorem and the Transfer Theorem yields the calculus for arbitrary polynomials in the Chern classes of $\Lambda_m(E)$.

3.1. **Flaglet geometry and the transfer theorem.** In this section we study the $(m, m-1)$ flag (or 'flaglet') Hilbert scheme, which we view as a correspondence between the Hilbert schemes for lengths $m$ and $m-1$ providing a way of transporting cycles, especially tautological ones, between these Hilbert schemes. We will make strong use of the results of [13].

Thus let

$$X^{[m,m-1]}_B \subset X^{[m]} \times_B X^{[m-1]}$$

denote the flag Hilbert scheme, parametrizing pairs of schemes $(z_1, z_2)$ satisfying $z_1 \supset z_2$. This comes equipped with a (flag) cycle map

$$c_{m,m-1} : X_B^{[m,m-1]} \to X_B^{(m,m-1)},$$

where $X_B^{(m,m-1)} \subset X_B^{(m)} \times_B X_B^{(m-1)}$ is the subvariety parametrizing cycle pairs $(c_m \geq c_{m-1})$. Note that the *normalization* of $X_B^{(m,m-1)}$ may be identified with $X_B^{(m-1)} \times_B X$; however the normalization map, though bijective, is not an isomorphism. Note also that we also have an ordered version $X_B^{\lceil m,m-1 \rceil}$, with its own cycle map

$$oc_{m,m-1} : X_B^{\lceil m,m-1 \rceil} \to X_B^m.$$

In addition to the obvious projections

(3.1.1)
$$\begin{array}{ccc} & X_B^{[m,m-1]} & \\ p_m \swarrow & & \searrow p_{m-1} \\ X_B^{[m]} & & X_B^{[m-1]} \end{array}$$

with respective generic fibres $m$ distinct points (corresponding to removing a point from a given $m$-tuple) and a generic fibre of $X/B$ (corresponding to adding a point to a given $m-1$-tuple), $X_B^{[m,m-1]}$ admits a natural map

(3.1.2)
$$a : X_B^{[m,m-1]} \to X,$$

$$(z_1 \supset z_2) \mapsto \mathrm{ann}(z_1/z_2)$$

(identifying $X$ with the Hilbert scheme of colength-1 ideals). Therefore $X_B^{[m,m-1]}$ admits a 'refined cycle map' (factoring the flag cycle map)

(3.1.3)
$$c : X_B^{[m,m-1]} \to X \times_B X_B^{(m-1)}$$

$$c = a \times (c_{m-1} \circ p_{m-1}).$$

Now in [13] (Theorem 5 et seq., especially Construction 5.4 p.442) we worked out a complete model for $X_B^{[m,m-1]}$, locally over $X_B^{(m,m-1)}$. Let

(3.1.4)
$$H_m \quad \subset \quad X_B^{(m)} \times \tilde{C}^m_{[u.,v.]} \subset X_B^{(m)} \times \mathbb{P}^{m-1}_{Z.},$$

(3.1.5)
$$H_{m-1} \quad \subset \quad X_B^{(m-1)} \times \tilde{C}^{m-1}_{[u'.,v'.]} \subset X_B^{(m-1)} \times \mathbb{P}^{m-2}_{Z'.}$$

be respective local models for $X_B^{[m]}, X_B^{[m-1]}$ as constructed in §1 above, with coordinates as indicated. Consider the subscheme

(3.1.6)
$$H_{m,m-1} \subset H_m \times_B H_{m-1} \times_{X_B^{(m)} \times X_B^{(m-1)}} X_B^{(m,m-1)}$$

defined by the equations

(3.1.7)
$$u'_i v_i = (\sigma_1^x - \sigma_1^{'x}) u_i v'_i, \quad v'_i u_{i+1} = (\sigma_1^y - \sigma_1^{'y}) v_{i+1} u'_i, \quad 1 \leq i \leq m-2$$

or alternatively, in terms of the $Z$ coordinates,

(3.1.8)
$$\begin{aligned} Z_iZ'_j &= (\sigma_1^x - \sigma_1^{'x})Z_{i+1}Z'_{j-1}, \quad i+1 \leq j \leq m-1 \\ &= (\sigma_1^y - \sigma_1^{'y})Z_{i-1}Z'_{j+1}, l \quad 1 \leq j \leq i-2. \end{aligned}$$

To 'explain' these relations in part, note that in the ordered model over $X_B^m$, we have

$$\sigma_1^x - \sigma_1^{'x} = x_m, \sigma_1^y - \sigma_1^{'y} = y_m$$

and then the analogue of (3.1.8) for the $G$ functions is immediate from (1.7.2). Then the result of [13], Thm. 5, is that $H_{m,m-1}$, with its map to $X_B^{(m,m-1)}$ is isomorphic to a neighborhood of the special fibre over $(mp, (m-1)p)$ of the flag Hilbert scheme $X_B^{[m,m-1]}$. In fact the result of [13] is even more precise and identifies $H_{m,m-1}$ with a subscheme of $H_m \times_B H_{m-1}$ and even of $H_{m-1} \times_B \tilde{C}^m \times_B X$, where the map to $X$ is the annihilator map $a$ above.

As noted in [13], Thm 5, the special fibre of the flag cycle map on $H_{m,m-1}$, aka the punctual flag Hilbert scheme, is a normal-crossing chain of $\mathbb{P}^1$'s:

(3.1.9)
$$C^{m,m-1} = \tilde{C}_1^m \cup \tilde{C}_1^{m-1} \cup \tilde{C}_2^m \cup \ldots \cup \tilde{C}_{m-2}^{m-1} \cup \tilde{C}_{m-1}^m \subset C^m \times C^{m-1}.$$

where the embedding is via

$$\tilde{C}_i^m \to C_i^m \times \{Q_i^{m-1}\}, \quad \tilde{C}_i^{m-1} \to \{Q_{i+1}^m\} \times C_i^{m-1}$$

and in particular,

(3.1.10)
$$\tilde{C}_i^m \cap \tilde{C}_i^{m-1} = \{(Q_{i+1}^m, Q_i^{m-1})\}, \tilde{C}_i^{m-1} \cap \tilde{C}_{i+1}^m = \{(Q_{i+1}^m, Q_{i+1}^{m-1})\}$$

where $Q_i^m = (x^{m-i+1}, y^i)$ as usual.

**Theorem 3.1.** *The cycle map $c_{m,m-1}$ exhibits the flag Hilbert scheme $X_B^{[m,m-1]}$ as the blow-up of the sheaf of ideals $\mathcal{I}_{D^{m,m-1}} := \mathcal{I}_{D^{m-1}}.\mathcal{I}_{D^m}$ on $X_B^{(m,m-1)}$.*

We shall not really need this result, just the explicit constructions above, so we just sketch the proof, which is analogous to that of Theorem 1.1. To begin with, it is again sufficient to prove the ordered analogue of this result, for the 'ordered flag cycle map'

$$X_B^{\lceil m,m-1 \rceil} \to X_B^m.$$

Here $X_B^{\lceil m,m-1 \rceil}$ is embedded as a subscheme of $X_B^{\lceil m \rceil} \times_{X_B^m} (X_B^{\lceil m-1 \rceil} \times_B X)$, and we have already observed that as such, it satisfies the equations (3.1.8).

Now we will use the following construction. Let $\mathcal{I}_1, \mathcal{I}_2$ be ideals on a scheme $Y$. Then the surjection of graded algebras

$$(\bigoplus_n \mathcal{I}_1^n) \otimes (\bigoplus_n \mathcal{I}_2^n) \to \bigoplus_n (\mathcal{I}_1\mathcal{I}_2)^n$$

yields a closed immersion

(3.1.11)
$$\mathrm{B}\ell_{\mathcal{I}_1\mathcal{I}_2} Y \hookrightarrow \mathrm{B}\ell_{\mathcal{I}_1} Y \times_Y \mathrm{B}\ell_{\mathcal{I}_2} Y;$$

the latter is in turn a subscheme of the Segre subscheme

$$\mathbb{P}(\mathcal{I}_1)\times_Y \mathbb{P}(\mathcal{I}_2)\subset \mathbb{P}(\mathcal{I}_1\otimes\mathcal{I}_2). \tag{3.1.12}$$

In our case, Theorem 1.1 allows us to identify

$$OH_m \simeq \mathrm{B}\ell_{\mathcal{I}_{OD^m}} X^m_B,\ OH_{m-1}\times_B X \simeq \mathrm{B}\ell_{\mathcal{I}_{OD^{m-1}.X^m_B}} X^m_B$$

(where $OH_m = H_m \times_{X_B^{(m)}} X^m_B$ etc.), whence an embedding

$$\mathrm{B}\ell_{\mathcal{I}_{OD^{m,m-1}}} X^m_B \to OH_m \times_{X^m_B} (OH_{m-1}\times_B X) \tag{3.1.13}$$

As observed above, the generators $G_i\cdot G'_j$ satisfy the analogues of the relations (3.1.8), so the image is actually contained in $OH_{m,m-1}$, so we have an embedding

$$\mathrm{B}\ell_{\mathcal{I}_{OD^{m,m-1}}} X^m_B \to OH_{m,m-1}. \tag{3.1.14}$$

We are claiming that this is an isomorphism. This can be verified locally, as in the proof of Theorem 1.1. □

One consequence of the explicit local model for $X_B^{[m,m-1]}$ is the following

**Corollary 3.2.** (i) *The projection $q_{m-1}$ is flat, with 1-dimensional fibres;*

(ii) *Let $z\in X_B^{[m-1]}$ be a subscheme of a fibre $X_s$, and let $z_0$ be the part of $z$ supported on nodes of $X_s$, if any. Then if $z_0$ is principal (i.e. Cartier) on $X_s$, the fibre $q_{m-1}^{-1}(z)$ is birational to $X_s$ and its general members are equal to $z_0$ locally at the nodes.*

*Proof.* (i) is proven in [12], and also follows easily from our explicit model $H_{m,m-1}$. As for (ii), we may suppose, in the notation of [13], that $z_0$ is of type $I_i^n(a)$. Now if $z'\in q_{m-1}^{-1}(z)$, then the part $z'_0$ of $z'$ supported on nodes must have length $n$ or $n+1$. In the former case $z'_0 = z_0$, while in the latter case $z'_0$ must equal $Q^{n+1}_{i+1}$ by [13], Thm. 5 p. 438, in which case $z'$ is unique, hence not general. □

Next we define the fundamental transfer operation. Essentially, this takes cycles from $X_B^{[m-1]}$ to $X_B^{[m]}$, but we also allow the additional flexibility of twisting by base classes via the $m$-th factor. Thus the *twisted transfer map* $\tau_m$ is defined by

$$\tau_m : A.(X_B^{[m-1]})\otimes A.(X)\to A.(X_B^{[m]})_{\mathbb{Q}}, \tag{3.1.15}$$

$$\tau_m = q_{m*}(q_{m-1}^*\otimes a^*).$$

Note that this operation raises dimension by 1 and preserves codimension. Suggestively, and a little abusively, we will write a typical decomposable element of the source of $\tau_m$ as $\gamma\beta_{(m)}$ where $\gamma\in A.(X_B^{[m-1]}), \beta\in A.(X)$. The following result which computes $\tau_m$ is a key to our inductive computation of Chern numbers.

**Theorem 3.3.** **(Tautological transfer)** *$\tau_m$ takes tautological classes on $X_B^{[m-1]}$ to tautological classes on $X_B^{[m]}$. More specifically we have, for any class $\beta\in A.(X)$:*

(i) *for any twisted diagonal class* $\gamma = \Gamma_{(n.)}[\alpha.^{(\lambda.)}]$,

(3.1.16) $$\tau_m(\gamma\beta_{(m)}) = \Gamma_{((n.)\coprod(1))}[\alpha.^{(\lambda.)}\coprod\beta];$$

(ii) *for any twisted node scroll* $F[\alpha] = F_j^{(n:n'.|n".)}(\theta)[\alpha', \alpha"]$,

(3.1.17) $$\tau_m(F[\alpha]\beta_{(m)}) = F_j^{(n:(n'.)\coprod(1)|n".))}[\alpha'\coprod\beta, \alpha"] + F_j^{(n:n'.|(n")\coprod(1))}[\alpha', \alpha"\coprod\beta]$$

(iii) *for any twisted node section* $\Gamma^{(m-1)}.F[\alpha]$ *with* $F[\alpha]$ *as above and*

$$n'. = (n_1^{(\mu_1')}, ...), n_1 > n_2 > ...,$$

*and ditto* $(n".)$, *we have*

(3.1.18)
$$\tau_m(\Gamma^{(m-1)}.F[\alpha]\beta_{(m)}) = \Gamma^{(m)}.\tau_m(F[\alpha]\beta_{(m)})$$
$$-\sum_\ell \mu_\ell' n_\ell' F_j^{(n:(n'.)^{-\ell}|n".)}(\theta)[\delta^*_{(n'.),\ell}(\alpha'\times\beta), \alpha"] - \sum_\ell \mu"_\ell n"_\ell F_j^{(n:(n'.)|n".^{-\ell})}(\theta)[\alpha', \delta^*_{(n".),\ell}(\alpha"\times\beta)]$$
$$- n\beta F_j^{(n+1:n'.|n".)}(\theta)[\alpha] - n\beta F_j^{(n+1:n'.|n".)}(\theta)[\alpha]$$

*where* $\delta_{(n.),\ell}$ *is as in* (0.2.8) *and where* $(n'.)^{+\ell}$ *is the distribution obtained from* $(n'.)$ *by replacing one block of size* $n'_\ell$ *with one of size* $n'_\ell + 1$.

*Proof.* Part (i) is obvious. As for Part (ii), the flatness of $q_{m-1}$ allows us to work over a general $z \in F$ and then Corollary 3.2, (ii) allows us to assume that the added point is a general point on the fibre $X_s$, which leads to (3.1.17).

As for (iii), note that on $X_B^{[m,m-1]}$, we can write

(3.1.19) $$q_m^*\Gamma^{(m)} = q_{m-1}^*\Gamma^{(m-1)} + \Delta^{(m)}$$

where $\Delta^{(m)}$ is the pullback from $X_B^{[m-1]} \times_B X$ of the locus of pairs $(z, w)$ where $w$ is a point subscheme of $z$. Restricted on $q_{m-1}^*F$, $\Delta^{(m)}$ remains an effective Cartier divisor which splits in two parts, depending on whether the point $w$ added to a scheme $z \in F$ is in the off-node or nodebound portion of $z$. It is easy to see that the first part gives rise to the 2nd and 3rd terms in the RHS of (3.1.18).

The analysis of the second part, which leads to the coefficient $n$ in the last two terms of (3.1.18), is a bit more involved. To begin with, it is easy to see that we may assume $m = n+1$, in which case $F$ is just a $\mathbb{P}^1$, namely $C_j^{m-1}$. Now as in (3.1.9),, the special fibre of the cycle map on $q_{m-1}^{-1}(C_j^{m-1})$, as a set, is given by $\tilde{C}_j^m \cup \tilde{C}_j^{m-1} \cup \tilde{C}_{j+1}^m$ and this coincides as a set with $\Delta^{(m)}.q_{m-1}^{-1}(C_j^{m-1})$. As $\tilde{C}_j^{m-1}$ collapses under $q_m$, the proof will be complete if we can show that the multiplicity of $\tilde{C}_j^m$ and $\tilde{C}_{j+1}^m$ on $\Delta^{(m)}.q_{m-1}^{-1}(C_j^{m-1})$ are both equal to $n = m-1$. We will do this for $\tilde{C}_{j+1}^m$ as the case of $\tilde{C}_j^m$ is similar and only notationally more cumbersome.

In that case, our assertion will be an elementary consequence of the equations on p. 440, l. 9-14 of [13], describing the local model $H_{m,m-1}$, as well as those on p. 433, describing the analogous local model $H_m$, to which equations we will be referring constantly in the remainder of the present proof. Note that $c_{m-i}$ (resp. $b'_{i-1}$) plays the role of the affine coordinate $u_i/v_i$ (resp. $v'_{i-1}/u'_{i-1}$). Also our $j+1$ is the $i$ there. We work on $q_{m-1}^{-1}(C_j^{m-1})$.

*Claim 1* : Over a neighborhood of $Q_{j+1}^{m-1}$, $q_{m-1}^{-1}(Q_{j+1}^{m-1})$ contains $\tilde{C}_{j+1}^m$ with multiplicity 1.

To see this note that the defining equations of $C_j^{m-1}$ on $X_B^{[m-1]}$ are given by setting all $a'_k$ and $d'_k$, as well as $c'_{m-i-1}$ to zero . By loc. cit. p.433 l.9, this implies that we have $b'_1 = ... = b'_{i-2} = 0$ on $q_{m-1}^{-1}(C_j^{m-1})$ as well. At a general point of $C_{j+1}^m$, $c_{m-i}$ is nonzero. Therefore we may consider $c_{m-i}$ as a unit. By loc. cit. p.440, eq. (15), we conclude $a_{m-i} = 0$. From this we see easily that all $a_k = d_k = 0$ except $d_{i-1}$, which is a local equation for $\tilde{C}_{j+1}^m$, while $b'_{i-1}$ is a coordinate along $C_j^{m-1}$ having $Q_{j+1}^{m-1}$ as its unique zero. Now by p.440 l. 14, $b'_{i-1}$ and $d_{i-1}$ differ by the multiplicative unit $-c_{m-i}$, therefore $b'_{i-1}$ generically cut out exactly $\tilde{C}_{j+1}^m$, which proves Claim 1.

*Claim 2* : The restriction of $\Delta^{(m)}$ on $\tilde{C}_j^{m-1}$ is the subscheme $((b'_{i-1})^n) = (d_{i-1}^n)$.

To prove Claim 2 we can pull back to the ordered version where the pullback of $\tilde{C}_j^{m-1}$ is totally ramified, hence can be written as $(m-1)U$, where $U$ maps isomorphically to $C_j^{m-1}$. On the other hand, on the ordered version of $q_{m-1}^{-1}(C_j^{m-1})$, we have

$$x_1 = ... = x_m = y_1 = ... = y_{m-1} = 0$$

(this by the vanishing of all the $a'_k, d'_k$, which are the elementary symmetric functions in $x_1, ...x_{m-1}$ and $y_1, ..., y_{m-1}$, respectively, and the vanishing of $a_{m-i} = x_1 + ... + x_m$). Therefore, the pullback of $\Delta^{(m)}$ is defined by the single nonzero generator, that is

$$\prod_{k=1}^{m-1}(y_m - y_k) = y_m^{m-1} \sim d_{i-1}^{m-1},$$

$d_{i-1}$ being a coordinate on $U \xrightarrow{\sim} C_j^{m-1}$. From this Claim 2 is obvious.

The conjunction of Claims 1 and 2 completes the proof of the Proposition. $\square$

3.2. **Full-flag transfer and Chern numbers.** We are now ready to tackle the computation of Chern numbers, and in fact all polynomials in the Chern classes of the tautological bundle on the relative Hilbert scheme $X_B^{[m]}$. The computation is based on passing from $X_B^{[m]}$ to the corresponding full-flag Hilbert scheme $W = W^m(X/B)$ studied in [12] and a *diagonalization theorem* for the total Chern class of (the pullback of) a tautological bundle on $W$, expressing it either as a simple (factorable) polynomial in diagonal classes induced from the various $X_B^{[n]}$, $n \leq m$,

plus base classes, or, more conveniently, as the product of the Chern class of a smaller tautological bundle and a diagonal class. Given this, we can compute Chern numbers essentially by repeatedly applying the transfer calculus of the last section.

We start by reviewing some results from [12]. Let

$$W^m = W^m(X/B) \xrightarrow{\pi^{(m)}} B$$

denote the relative flag-Hilbert scheme of $X/B$, parametrizing flags of subschemes

$$z. = (z_1 < ... < z_m)$$

where $z_i$ has length $i$ and $z_m$ is contained in some fibre of $X/B$. Let

$$w^m : W^m \to X_B^{[m]}, w^{m,i}. : W^m \to X_B^{[i]}$$

be the canonical (forgetful) maps. Let

$$a_i : W^m \to X$$

be the canonical map sending a flag $z.$ to the 1-point support of $z_i/z_{i-1}$ and

$$a^m = \prod a_i : W^m \to X_B^m$$

their (fibred) product, which might be called the 'ordered cycle map'. Let

$$\mathcal{I}_m < \mathcal{O}_{X_B^{[m]} \times_B X}$$

be the universal ideal of colength $m$. For any coherent sheaf on $X$, set

$$\lambda_m(E) = p_{X_B^{[m]}*}(p_X^*(E) \otimes (\mathcal{O}_{X_B^{[m]} \times_B X}/\mathcal{I}_m))$$

These are called the *tautological sheaves* associated to $E$; they are locally free if $E$ is. Abusing notation, we will also denote by $\lambda_m(E)$ the pullback of the tautological sheaf to appropriate flag Hilbert schemes mapping naturally to $X_B^{[m]}$, such as $W^m$ or $X_B^{[m,m-1]}$. With a similar convention, set

$$\Delta^{(m)} = \Gamma^{(m)} - \Gamma^{(m-1)}. \tag{3.2.1}$$

The various tautological sheaves form a flag of quotients on $W^m$:

$$...\lambda_{m.i}(E) \twoheadrightarrow \lambda_{m,i-1}(E) \twoheadrightarrow ... \tag{3.2.2}$$

This flag makes possible a simple formula for the total Chern class of the tautological bundles, namely the following *diagonalization theorem* ( [12], Cor. 3.2):

**Theorem 3.4.** *The total Chern class of the tautological bundle $\lambda_m(E)$ is given by*

$$c(\lambda_m(E)) = \prod_{i=1}^{m} c(a_i^*(E)(-\Delta^{(i)})) \tag{3.2.3}$$

An analogue of this, more useful for our purposed, holds already on the flaglet Hilbert scheme. It can be proved in the same way, or as an easy consequence of Thm 3.4

**Corollary 3.5.** *We have an identity in* $A.(X_B^{[m,m-1]})_{\mathbb{Q}}$*:*

(3.2.4) $$c(\lambda_m(E)) = c(\lambda_{m-1}(E))c(a_m^*(E)(-\Delta^{(m)})).$$

*Proof.* By Theorem 3.4, the RHS and LHS pull back to the same class in $W^m$. As the projection $W^m \to X_B^{[m,m-1]}$ is generically finite, they agree mod torsion. □

Motivated by this result we make the following definition.

**Definition 3.6.** *Let $R$ be a $\mathbb{Q}$-subalgebra of $A(X)$ containing the canonical class $\omega$. The Chern tautological ring on $X_B^{[m]}$, denoted*

$$TC_R^m = TC_R^m(X/B),$$

*is the $R$-subalgebra of $A(X_B^{[m]})_{\mathbb{Q}}$ generated by the Chern classes of $\lambda_m(E)$ and the discriminant class $\Gamma^{(m)}$.*

*Remark* 3.7. If $E$ is a line bundle, then it is easy to see from Theorem 3.4 that

$$c_1(\lambda_m(E)) = mc_1(E) - \Gamma^{(m)}. \quad \square$$

The following is the main result of this paper.

**Theorem 3.8.** *There is a computable inclusion*

(3.2.5) $$TC_R^m \to T_R^m.$$

*More explicitly, any polynomial in the Chern classes of $\lambda_m(E)$, in particular the Chern numbers, can be computably expressed as a linear combination of standard tautological classes: twisted diagonal classes, twisted node scrolls, and twisted node sections.*

*Proof.* For $m = 1$ the statement is essentially vacuous. For $m = 2$ it is a consequence of the Module Theorem 2.1. For general $m$, we assume inductively the result is true for $m - 1$. Given any polynomial $P$ in the Chern classes of $\lambda_m(E)$, Corollary 3.5 implies that we can write its pullback on $X_B^{[m,m-1]}$ as a sum of terms of the form $p^*_{X_B^{[m-1]}}Q.(\Gamma^{(m)})^k.S$ where $Q \in TC_R^{m-1}$. By induction, $Q \in T_R^{m-1}$, so by the Transfer Theorem 3.3, $\tau_m(Q) \in T_R^m$. By the projection formula and the Module Theorem 2.1, it follows that $P \in T_R^m$. □

*Remark* 3.9. This result suggests the natural question: is $T_R^m$ a ring? more ambitiously, is the inclusion $TC_R^m \to T_R^m$ an equality?

3.3. **Example: trisecants to one space curve curve.** If $X$ is a smooth curve of degree $d$ and genus $g$ in $\mathbb{P}^3$, the virtual degree of its trisecant scroll, i.e. the virtual number of trisecant lines to $X$ meeting a generic line, is given by $c_3(\bigwedge^2(\Lambda_3(\mathcal{O}_X(1))),$ which can be easily computed to be

$$\frac{1}{6}(2d^3 - 12d^2 + 16d - 3d(2g-2) + 6(2g-2)) \tag{3.3.1}$$

3.4. **Example: trisecants in a pencil.** With $X/B$ as above ($B$ a smooth curve), suppose

$$f : X \to \mathbb{P}^{2m-1}$$

is a morphism. One, quite special, class of examples of this situation arises as what we call a *generic rational pencil*; that is, generally, the normalization of the family of rational curves of fixed degree $d$ in $\mathbb{P}^r$ (so $r = 2m-1$ here) that are incident to a generic collection $A_1, ...A_k$ of linear spaces, with

$$(r+1)d + r - 4 = \sum(\text{codim}(A_i) - 1);$$

see [10] and references therein, or [9] for an 'executive summary'. While our result seems new in this case, we note that it applies to curves of arbitrary genus.

Returning to the general situation, one expects a finite number $N_m$ of curves $f(X_b)$ to admit an m-secant $(m-2)$-plane, and this number can be evaluated as follows. Let $G = G(m-1, 2m)$ be the Grassmannian of $(m-2)$-planes in $\mathbb{P}^{2m-1}$, with rank-$(m+1)$ tautological subbundle $S$, and let $L = f^*\mathcal{O}(1)$. Then

$$m!N_m = \int\limits_{W^m \times G} c_{m(m+1)}(S^* \boxtimes w^*\lambda_m(L))$$

$$= \int\limits_{W^m \times G} c_{m+1}(S^*(L_{(1)}))c_{m+1}(S^*(L_{(2)} - \Delta^{(2)})) \cdots c_{m+1}(S^*(L_{(m)} - \Delta^{(m)}))$$

$$= \int\limits_{W^m \times G} \prod_{i=1}^{m}(\sum_{j=0}^{m+1} \binom{m+1}{j} c_{m+1-j}(S^*)(L^{(i)} - \Delta^{(i)})^j)$$

$$= \sum_{|(j.)|=m+1} \int\limits_G c_{m+1-j_1,...,m+1-j_m}(S^*) \int\limits_{W^m} (L_{(1)})^{j_1}(L_{(2)} - \Delta^{(2)})^{j_2} ... (L_{(m)} - \Delta^{(m)})^{j_m}$$

where $c_{u,v,w...} = c_u c_v c_w \cdots$ and, applied to $S^*$, represents the condition that an $(m-2)$-plane in $\mathbb{P}^{2m-1}$ meet a generic collection of planes of respective dimensions $u, v, w, ...$ Note that only terms with $j_1 + ... + j_k \leq k+1, \forall k$, can contribute. By the intersection calculus developed above, this number can be computed in terms of the characters

$$b = L^2, d = \deg_\pi(L), \omega^2, \sigma, \omega.L, \deg_\pi(\omega) = 2g-2, g = \text{fibre genus};$$

in the generic rational pencil case, all these characters can be computed by recursion on $d$.

Suppose now that $m=3$, where the only relevant $(j.)$ are

$$(2,1,1),(1,1,2),(2,0,2),(1,2,1),(1,0,3),(0,3,1),(0,2,2),(0,1,3),(0,0,4).$$

In each of these cases, it is easy to see that the $G$ integral evaluates to 1. The $W$ integrals may be evaluated by the calculus developed above. The general procedure is to proceed inductively, each time transferring the leftmost factor from the $X_B^{[i]}$ from whence it came to $X_B^{[i+1]}$. We will repeatedly be using Corollaries 2.11 and 2.13, as well as standard projection formulas (for the symmetrization map). After the first transfer (to $X_B^{[2]}$, we will treat the resulting term as polynomial in $\Gamma^{(3)}$ and break things up according to the power of $\Gamma^{(3)}$ involved. The main multiplication rules to be used are the following. We will use the notation $[\alpha,...]$, for a base class $\alpha$, in place of $\Gamma_{(1.)}[\alpha,...]$, where $(1.)$ is a trivial (singleton blocks only) distribution. We also recall that $\Gamma^{(3)}[\alpha]$ is short for $\Gamma^{(3)}[\alpha,1]$).

**Multiplication rules**

(i)

$$[\alpha,\beta,\gamma].\Gamma^{(3)}=2(\Gamma^{(3)}[\alpha.\beta,\gamma]+\Gamma^{(3)}[\alpha.\gamma,\beta]+\Gamma^{(3)}[\beta.\gamma,\alpha])$$

(ii)

$$\Gamma^{(3)}[\alpha,\beta].\Gamma^{(3)}=\Gamma_{(3)}[\alpha.\beta]-\Gamma^{(3)}[\alpha.\omega,\beta]$$

┘

The detailed computation follows.

(2,1,1) that is, $L^2_{(1)}(L_{(2)}-\Delta^{(2)})(L_{(3)}-\Delta^{(3)})$. First,

$$\tau_2(L^2_{(1)}(L_{(2)}-\Delta^{(2)}))=[L^2,L]-2\Gamma^{(2)}[L^2].$$

Next,

$$\tau_3(([L^2,L]-2\Gamma^{(2)}[L^2])L_{(3)}=[L^2,L^{(2)}]-2\Gamma^{(3)}[L^2,L]=bd^2-bd,$$

$$\tau_3([L^2,L]-2\Gamma^{(2)}[L^2]).\Gamma^{(3)}=4\Gamma^{(3)}[L^2,L]-2\Gamma_{(3)}[L^2]=2bd-2b$$

Thus the total is $bd^2-3bd+2b=\boxed{b(d-1)(d-2)}$.

<u>(1,1,2)</u> : We treat this as a polynomial in $\Gamma^{(3)}$. The terms are as follows. Degree 0:

$$\tau_3(\tau_2(L_{(1)}(L_{(2)}-\Gamma^{(2)})(\Gamma^{(2)})^2)+2\tau_2(L_{(1)}(L_{(2)}-\Gamma^{(2)})\Gamma^{(2)})L_{(3)}+\tau_2(L_{(1)}(L_{(2)}-\Gamma^{(2)}))L_{(3)}^2)=$$

(as the $(\Gamma^{(2)})^2$ doesn't contribute for dimension reasons)

$$4\Gamma^{(3)}[L^2,L]+4\Gamma^{(3)}[L.\omega,L]+[L^{(2)},L^2]-2\Gamma^{(3)}[L,L^2]=2bd+2dL.\omega+bd^2-bd=$$
$$bd+bd^2+2dL.\omega$$

degree 1: similarly

$$-2\tau_3(2\Gamma^{(2)}[L^2]+2\Gamma^{(2)}[L.\omega]+[L^{(2)}]L_{(3)}-2\Gamma^{(2)}[L]L_{(3)})\Gamma^{(3)}=$$
$$-2(2\Gamma^{(3)}[L^2]+2\Gamma^{(3)}[L.\omega]+[L^{(3)}]-2\Gamma^{(3)}[L,L])\Gamma^{(3)}=$$
$$-2(2\Gamma_{(3)}[L^2]+2\Gamma_{(3)}[L.\omega]+6\Gamma^{(3)}[L^2,L]-2\Gamma_{(3)}[L^2]+2\Gamma^{(3)}[L.\omega,L])=$$
$$-2(2b+2L.\omega+3bd-2b+dL.\omega)=-2(2L.\omega+3bd+dL.\omega)$$

degree 2:

$$([L^{(2)}]-2\Gamma^{(3)}[L])(\Gamma^{(3)})^2=(2\Gamma^{(3)}[L^2]+4\Gamma^{(3)}[L,L]-2\Gamma_{(3)}[L]+2\Gamma^{(3)}[L.\omega])\Gamma^{(3)}=$$
$$2\Gamma_{(3)}[L^2]+4\Gamma_{(3)}[L^2]-4\Gamma^{(3)}[L.\omega,L]+6\Gamma_{(3)}[L.\omega]+2\Gamma_{(3)}[L.\omega]=6b-2dL.\omega+8L.\omega$$

total: $\boxed{-5bd+bd^2+6b-2dL.\omega+4L.\omega}$

<u>(2,0,2)</u> This case is similar to (1,1,2) and easier. The result is

$$\boxed{-2bd-b(2g-2)+2b.}$$

<u>(1,2,1)</u> This is again quite similar to the (2,1,1) case treated above, and yields

$$\boxed{(bd-2b-L.\omega)(d-2)}.$$

<u>(1,0,3)</u> Again we consider this as a polynomial in $\Gamma^{(3)}$ and compute term by term. For degree 0 we have

$$\tau_3(6\Gamma^{(2)}[L](L_{(3)})^2-6\Gamma^{(2)}[L.\omega]L^{(3)})=3bd-3dL.\omega$$

degree 1:

$$-3([L,L^2]+4\Gamma^{(3)}[L,L]-2\Gamma^{(3)}[\omega.L])\Gamma^{(3)}=$$
$$-3(2\Gamma^{(3)}[L,L^2]+2\Gamma^{(3)}[L^2,L]+4\Gamma_{(3)}[L^2]-4\Gamma^{(3)}[L.\omega,L]-2\Gamma_{(3)}[\omega.L])=$$
$$-3(bd+bd+4b-2d\omega.L-2\omega.L)$$

For degree 2:

$$3([L^{(2)}]+2\Gamma^{(3)}[L])(\Gamma^{(3)})^2=3(2\Gamma^{(3)}[L^2]+4\Gamma^{(3)}[L,L]+2\Gamma_{(3)}[L]-2\Gamma^{(3)}[\omega.L])\Gamma^{(3)}=$$
$$3(2\Gamma_{(3)}[L^2]+4\Gamma_{(3)}[L^2]-4\Gamma^{(3)}[\omega.L,L]+6\Gamma_{(3)}[\omega.L]-2\Gamma_{(3)}[\omega.L])=$$
$$3(2b+4b-2d\omega.L-6\omega.L-2\omega.L)$$

For degree 3: by Corollary 2.13, we get

$$24\omega.L - d\omega^2$$

Summing up, we get $\boxed{-3bd - 3dL.\omega + 6b + 6L.\omega - d\omega^2}$

<u>(0,3,1)</u> Expanding, we get

$$\tau_2((L_{(2)} - \Delta^{(2)})^3) = -6\Gamma^{(2)}[L^2] - 6\Gamma^{(2)}[\omega.L] - 2(\Gamma^{(2)})^3 = -3b - 3\omega.L - (\omega^2 - \sigma)$$

Since this is a point cycle, multiplying by $L_{(3)} - \Delta^{(3)}$ or $L^{(3)} - \Gamma^{(3)}$ again just multiplies the coefficient by $d-2$, for a total of $\boxed{(-3b - 3L.\omega - (-\sigma + \omega^2))(d-2)}$.

<u>(0,2,2)</u> Again working degree by degree in $\Gamma^{(3)}$, we get:

degree 0:

$$\begin{aligned}\tau_3(2\tau_2((L_{(2)}^2 - 2L_{(2)}\Gamma^{(2)} + (\Gamma^{(2)})^2)\Gamma^{(2)})L_{(3)}) + \tau_3(\tau_2(L_{(2)}^2 - 2L_{(2)}\Gamma^{(2)} + (\Gamma^{(2)})^2)(L_{(3)})^2) = \\ 4\Gamma^{(3)}[L^2, L] + 8\Gamma^{(3)}[L.\omega, L] + [\omega^2 - \sigma, L] - 4\Gamma^{(3)}[L, L^2] - 2\Gamma^{(3)}[\omega, L^2] = \\ 4d\omega.L + 2d(\omega^2 - \sigma) - b(2g-2)\end{aligned}$$

degree 1:

$$\begin{aligned}-2\tau_3(\tau_2((L_{(2)}^2 - 2L_{(2)}\Gamma^{(2)} + (\Gamma^{(2)})^2)\Gamma^{(2)}) + \tau_2(L_{(2)}^2 - 2L_{(2)}\Gamma^{(2)} + (\Gamma^{(2)}))L_{(3)})\Gamma^{(3)} = \\ -2(2\Gamma^{(3)}[L^2] + 4\Gamma^{(3)}[\omega.L] + [\omega^2 + \sigma] + [L^2, L] - 4\Gamma^{(3)}[L, L] - 2\Gamma^{(3)}[\omega, L])\Gamma^{(3)} = \\ -2(2b + 4\omega.L + 2(\omega^2 - \sigma) + 2bd - 4b + 2d\omega.L - 2\omega.L + 2d\omega^2) = \\ 4b - 4\omega.L - 4(\omega^2 - \sigma) - 4bd - 4d\omega.L - 2d\omega^2\end{aligned}$$

degree 2:

$$\begin{aligned}([L^2] - 4\Gamma^{(3)}[L] - 2\Gamma^{(3)}[\omega] + F_1^{(2:1|0)} + F_1^{(2:0|1)})(\Gamma^{(3)})^2 = \\ (4\Gamma^{(3)}[L^2] + 2\Gamma^{(3)}[1, L^2] - 4\Gamma_{(3)}[L] + 4\Gamma^{(3)}[L.\omega] - 2\Gamma_{(3)}[\omega] + 2\Gamma^{(3)}[\omega^2])\Gamma^{(3)} - 6\sigma = \\ 4\Gamma_{(3)}[L^2] + 2\Gamma_{(3)}[L^2] - 2\Gamma^{(3)}[\omega, L^2] + 12\Gamma_{(3)}[L.\omega] + 4\Gamma_{(3)}[L.\omega] + 6\Gamma_{(3)}[\omega^2] + 2\Gamma_{(3)}[\omega^2] - 6\sigma = \\ = 4b + 2b - b(2g-2) + 12L.\omega + 4L.\omega + 6\omega^2 + 2\omega^2 - 6\sigma = 6b - b(2g-2) + 16\omega.L + 8\omega^2 - 6\sigma\end{aligned}$$

The total is $\boxed{-2d\sigma + 10b + 12\omega.L + 4\omega^2 - 2\sigma - 4bd - 2b(2g-2)}$

<u>(0,1,3)</u> Again we consider this as polynomial in $\Gamma^{(3)}$. The terms are:

degree 0:

$$\begin{aligned}\tau_3((L_{(2)} - \Gamma^{(2)})(3L_{(3)}^2\Gamma^{(2)} + 3L_{(3)}(\Gamma^{(2)})^2) = 6\Gamma^{(3)}[L, L^2] + 6\Gamma^{(3)}[\omega, L^2] - 6\Gamma^{(3)}[\omega.L, L] - 3[(\omega^2 - \sigma), L] = \\ 3db + 3b(2g-2) - 3d\omega.L - 3d(\omega^2 - \sigma)\end{aligned}$$

degree 1:

$$
\begin{aligned}
-3\tau_3((L_{(2)}-\Gamma^{(2)})(L_{(3)}^2+2L_{(3)}\Gamma^{(2)}+(\Gamma^{(2)})^2))\Gamma^{(3)} &= \\
3(-[L,L^2]+2\Gamma^{(3)}[1,L^2]-4\Gamma^{(3)}[L,L]+4\Gamma^{(3)}[\omega,L]-2\Gamma^{(3)}[\omega.L]+[\omega^2-\sigma])\Gamma^{(3)} &= \\
3(-2bd-b(2g-2)-2b+2d\omega.L+2d\omega^2+2\omega.L+2(\omega^2-\sigma))&
\end{aligned}
$$

degree 2:

$3\tau_3((L_{(2)}-\Gamma^{(2)})(L_{(3)}+\Gamma^{(2)}))(\Gamma^{(3)})^2 =$
$3([L,L]-2\Gamma^{(3)}[1,L]+2\Gamma^{(3)}[L]+2\Gamma^{(3)}[\omega])(\Gamma^{(3)})^2 =$
$3((2\Gamma^{(3)}[L^2]+4\Gamma^{(3)}[L,L]+2\Gamma^{(3)}[\omega,L]-2\Gamma_{(3)}[L]-2\Gamma^{(3)}[L.\omega]+2\Gamma_{(3)}[L]-2\Gamma^{(3)}[\omega^2]+2\Gamma_{(3)}[\omega])\Gamma^{(3)} =$
$3(2\Gamma_{(3)}[L^2]-4\Gamma^{(3)}[L.\omega,L]+4\Gamma_{(3)}[L^2]-2\Gamma^{(3)}[\omega^2,L]+2\Gamma_{(3)}[\omega.L]+6\Gamma_{(3)}[L.\omega]-2\Gamma_{(3)}[L.\omega]-6\Gamma_{(3)}[L.\omega]$
$-2\Gamma_{(3)}[\omega^2]-6\Gamma_{(3)}[\omega^2]) = 3(6b-2dL.\omega-d\omega^2-8\omega^2)$

degree 3:

$-\tau_3(L_{(2)}-\Gamma^{(2)})(\Gamma^{(3)})^3 = -([L]-2\Gamma^{(3)})(\Gamma^{(3)})^3 =$

(using 2.13 and 2.30)

$= 12L.\omega-d\omega^2+26\omega^2+2(-6\sigma-3\sigma)+d\sigma$

Total: $\boxed{-3db-3d\omega.L-d\omega^2+4d\sigma+12b+18\omega.L+8\omega^2-24\sigma}$

(0,0,4) Proceeding as above, we get: degree 0:

$$
\begin{aligned}
\tau_3(12(\Gamma^{(2)})^2(L_{(3)})^2+8(\Gamma^{(2)})^3L_{(3)}) = -12\Gamma^{(3)}[\omega,L^2]+8\Gamma^{(3)}[\omega^2,L]+4[\omega^2-\sigma,L] &= \\
-6b(2g-2)+4d\omega^2+4d(\omega^2-\sigma).&
\end{aligned}
$$

degree 1:

$$
\begin{aligned}
-4\tau_3((\Gamma^{(2)}+L_{(3)})^3)\Gamma^{(3)} = -4\tau_3(3\Gamma^{(2)}L_{(3)}^2+3(\Gamma^{(2)})^2L_{(3)}+(\Gamma^{(2)})^3)\Gamma^{(3)} &= \\
-4(6\Gamma^{(3)}[1,L^2]-6\Gamma^{(3)}[\omega,L]+[\omega^2-\sigma])\Gamma^{(3)} &= \\
-4(6\Gamma_{(3)}[L^2]-6\Gamma^{(3)}[\omega,L^2]-6\Gamma_{(3)}[\omega.L]+6\Gamma^{(3)}[\omega^2,L]+2[\omega^2-\sigma]) &= \\
-24b+12b(2g-2)+24\omega.L-8\omega^2+8\sigma&
\end{aligned}
$$

degree 2:

$$+6\tau_3((L_{(3)}+\Gamma^{(2)})^2)(\Gamma^{(3)})^2 = 6\tau_3(L_{(3)}^2+2L_{(3)}.\Gamma^{(2)}+(\Gamma^{(2)})^2)(\Gamma^{(3)})^2 =$$
$$6([L^2]+4\Gamma^{(3)}[L]-2\Gamma^{(3)}[\omega]+\frac{1}{2}(F_1^{(2:1|0)}+F_1^{(2:0|1)}))(\Gamma^{(3)})^2 =$$
$$6(4\Gamma^{(3)}[L^2]+2\Gamma^{(3)}[1,L^2]+4\Gamma_{(3)}[L]-4\Gamma^{(3)}[L.\omega]-2\Gamma_{(3)}[\omega]+2\Gamma^{(3)}[\omega^2]+\frac{1}{2}(F_1^{(2:1|0)}+F_1^{(2:0|1)})(\Gamma^{(3)}))\Gamma^{(3)} =$$
$$6(4\Gamma_{(3)}[L^2]+2\Gamma_{(3)}[L^2]-2\Gamma^{(3)}[\omega,L^2]-12\Gamma_{(3)}[L.\omega]-4\Gamma_{(3)}[L.\omega]+6\Gamma_{(3)}[\omega^2]+2\Gamma_{(3)}[\omega^2]-3\sigma) =$$
$$6(4b+2b-b(2g-2)-12L.\omega-4L.\omega+6\omega^2+6\omega^2+3\sigma) = 6(6b-b(2g-2)-16L.\omega+8\omega^2-3\sigma)$$

degree 3:

$$-4\tau_3(\Gamma^{(2)}+L_{(3)})(\Gamma^{(3)})^3 = -4(2\Gamma^{(3)}+[L])(\Gamma^{(3)})^3 =$$
$$-4(-24\Gamma_{(3)}[\omega.L]+2\Gamma^{(3)}[\omega^2,L]+2(\Gamma^{(3)})^4) = -4(-24\omega.L+d\omega^2+2(12\Gamma_{(3)}[\omega^2]+\Gamma_{(3)}[\omega^2]-6\sigma-3\sigma) =$$
$$-4(-24\omega.L+d\omega^2-d\sigma+26\omega^2-18\sigma)$$

degree 4:

$$\tau_3(\tau_2(1))(\Gamma^{(3)})^4 = 6(13\omega^2-9\sigma)$$

Total: $\boxed{12b+24\omega.L+14\omega^2}$

**Grand total**:

$\boxed{(3d^2-25d+60)b+(-12d+72)L.\omega+(-3d+28)\omega^2-3b(2g-2)+(3d-20)\sigma}$

This formula has been obtained by other means by Ethan Cotterill [2]

3.5. **Example: double points.** Let $X/B$ be an arbitrary nodal family and $f: X \to \mathbb{P}^n$ a morphism. Consider the relative double points of $f$, i.e. double points on fibres. This locus is given on $X_B^{[2]}$ as the degeneracy locus of a bundle map

$$\phi: (n+1)\mathcal{O} \to \Lambda_2(L), L := f^*\mathcal{O}(1).$$

By Porteus, the virtual fundamental class of this locus is given by the Segre class $s_n(\Lambda_2(L)^*)$, which equals

(3.5.1) $$\sum_{i=0}^{n}(L_1)^{n-i}(L_2-\Gamma)^i, \Gamma = \Gamma^{[2]}.$$

The powers of $\Gamma$ can be evaluated using Corollary 2.29. Pushing the result down to $X_B^2$ for simplicity yields

$$\sum_{i=0}^{n} L_1^{n-i}L_2^i - \sum_{i=0}^{n} L_1^{n-i}(\sum_{j=1}^{i}(\Gamma[\omega^{j-1}]+\sum_{s,k}\delta_{s*}(\psi_x^{j-2-k}\psi_y^k))L^{i-j})$$

To describe the direct image of this on $B$, we need some notation. Recall that $\kappa_j = \pi_*(\omega^{j+1})$. Extending this, we may set

$$\kappa_j(L) = \pi_*(L^{j+1}), \kappa_{i,j}(L, M) = \pi_*(L^{i+1}M^{j+1}). \tag{3.5.2}$$

Note that in our case $\kappa_j(L)$ may be interpreted as the class of the locus of curves meeting a generic $\mathbb{P}^{n-j}$. Also, for each boundary datum $(T_s, \delta_s, \theta_s)$, $T_s$ admits a map to $\mathbb{P}^n$ via either the $x$ or $y$-section (the two maps are the same), via which we can pull back $L^j$, which corresponds to the locus of boundary curves whose node $\theta_s$ meets $\mathbb{P}^{n-j}$. Then pushing the above down to $B$ yields

$$2m_2 = (-1)^n(\sum_{i=1}^{n-1} \kappa_{i-1}(L)\kappa_{n-i-1}(L) - \kappa_{n-j-1,j-2}(L, \omega) + \sum_{s,k} \delta_{s*}(L^{n-j}\psi_x^{j-2-k}\psi_y^k)) \tag{3.5.3}$$

More generally, for any smooth variety $Y$ of dimension $n$ and map $f : X \to Y$, one can use the double-point formula of [12], Th. 3.3ter, p. 1208, to evaluate the class of the double-point locus in $X_B^2$ in terms of the diagonal class $\Delta_Y$ on $Y \times Y$ as

$$\begin{aligned} 2m_2 &= (f^2)^*(\Delta_Y) + \sum_{i\geq 1}(-\Gamma^i)c_{n-i}(T_Y) \\ &= (f^2)^*(\Delta_Y) + \sum_{i\geq 1}(-\Gamma[\omega^{i-1}] + \frac{1}{2}\sum_{s,j}\delta_{s*}(\psi_x^{i-j-3}\psi_y^j))c_{n-i}(T_Y) \end{aligned} \tag{3.5.4}$$

Applying this set-up to the case $L = \omega$, one would like in principle to be able to compute fundamental classes of loci of hyperelliptics (and more generally, $\mathfrak{M}_d^r$ loci in $\overline{\mathfrak{M}}_g$). The problem is that the naive notion of canonical curve in $\mathbb{P}^{g-1}$ is ill-behaved over the boundary and requires substantial modification there. This work is currently in progress.

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